Well-posed systems-The LTI case and beyond

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Abstract

This survey is an introduction to well-posed linear time-invartiant (LTI) systems for non-specialists. We recall the more general concept of a system node, classical and generalized solutions of system equations, criteria for well-posedness, the subclass of regular linear systems, some of the available linear feedback theory. Motivated by physical examples, we recall the concepts of impedance passive and scattering passive systems, conservative systems and systems with a special structure that belong to these classes. We illustrate this theory by examples of systems governed by heat and wave equations. We develop local and global well-posedness results for LTI systems with nonlinear (in particular, bilinear) feedback, by extracting the abstract idea behind various proofs in the literature. We apply these abstract results to derive well-posedness results for the Burgers and Navier-Stokes equations.

Key words: Well-posed linear system, operator semigroup, regular linear system, impedance passive system, heat equation, scattering passive system, scattering conservative system, wave equation, non-linear feedback, Burgers equation, local well-posedness, Navier-Stokes equations.

1 Overview

This is a survey about well-posed systems, intended for newcomers to the field. Thus, no prior background on well-posed systems is assumed and we intend to guide the reader through the many concepts and available results, explaining their origin and significance as best as we can. We assume that the reader has a basic understanding of functional analysis and operator semigroups.

Informally speaking, a system is *well-posed* if on any time interval $[\tau, t]$, for any initial state x_0 in the state space and any input function u in a specified space of functions, it has a unique state trajectory x and a unique output function y, both defined on $[\tau, t]$. Moreover, y must belong to a specified space of functions, and both $x(t)$ and y must depend continuously on $x(\tau)$ and on u. This concept is general and can be made precise for many classes of non-linear and/or time-varying systems. However, most attention in the literature has been devoted to the simplest particular case, namely, linear and time-invariant (LTI) systems, because here we have strong tools to develop the theory. In the LTI context, if the state space is finite-dimensional, then well-posedness is not an issue and is usually not even mentioned. The theory focuses on systems with an infinite-dimensional state space, usually a Hilbert space. This is motivated by a variety of systems described by partial differential or delay equations, that can be shown to fit into this framework. Establishing well-posedness is usually not a goal in itself but opens the way for dealing with control and/or estimation problems by trying to mimic the rich finite-dimensional control theory using "operators in place of matrices" at the conceptual level. It is not easy and it does not always work, but we keep trying.

There are now two books available on well-posed linear systems: the monograph of Olof Staffans [65] and the recent graduate lectures of Birgit Jacob and Hans Zwart [38] (actually on a different but closely related topic). It is not easy to write a survey "in the shadow" of these two excellent books. We hope that our emphasis on extensions, as well as our somewhat different point of view will be deemed a useful addition to the literature.

The authors together with Olof Staffans have published

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the survey [83] with a similar topic in 2000. So what can justify this new survey? We hope that the following will count: (1) In the 14 years that passed, there have been many new results and developments, shifts in emphasis, and we have learned a few new tricks. (2) In the survey [83], the emphasis was on conservative linear systems, which was an exciting new topic at that time. The survey was strictly about LTI systems only, and the same is true about the books [65] and [38] mentioned earlier. Here, we go beyond this framework by exporing well-posedness results for well-posed linear systems with nonlinear feedback. Most results will be given without proof (with citations) but we also include several new results, and for those of course we give the proof.

This survey *does not* cover the following topics: exact controllability and exact observability, stability and stabilization, optimal control and optimal estimation. Indeed, these topics are not directly related to well-posedness (even though they use results about it). To our regret, because of length constraints, we also have to leave out topics that would have been very well suited in this paper. One such topic is the Lax-Phillips semigroup associated to a well-posed system (the connection between well-posed systems theory and scattering theory), for which we refer to [7,9,59,65,67,68]. Another topic that we are compelled to leave out are the time-varying well-posed linear systems, for which we refer to [12,33,58,59] (an incomplete list).

It will be easier to follow the more abstract developments in the later sections of this paper if we first introduce the main concepts (system equations, linearity, time-invariance, well-posedness, impedance and scattering passivity) and some of the results in the finite-dimensional context. This is our aim in Section 2.

In Section 3 we give a brief overview of the main facts known about well-posed linear time-invartiant systems in the Hilbert space context. We give the motivation and introduce the concept, after which we discuss the representation of such systems via a semigroup generator A, a control operator B, an observation operator C and a transfer function $\mathbf{G}$. We recall the admissibility concepts for B and C.

In Section 4 we introduce the larger class of LTI systems known as system nodes. This is a simple and very useful concept when we model physical systems, or when we introduce special classes of systems, as there are almost no well-posedness assumptions involved, and well-posedness can be checked at a later stage. We introduce the concepts of classical and generalized solution of the system equations, and discuss their properties, in particular in the well-posed case.

In Section 5 we introduce regular linear systems, a subclass of the well-posed ones for which there is a well-defined feedthrough operator, that expresses the instantaneous effect of the input signal on the output signal. The feedthrough operator (if it exists) is the strong limit of the transfer function at $+\infty$. We recall different equivalent ways to express regularity, and a simpler way to to write the system equations. We give several examples of regular systems from the literature, including a wave and a plate equation. We also recall the basic facts of the linear feedback theory developed for well-posed (and in particular, for regular) linear systems.

In Section 6 we introduce impedance passive system nodes, scattering passive (hence well-posed) linear systems and scattering conservative systems. We explain how systems with certain special structures (encountered in mathematical physics) belong to these special classes. We give examples of systems with these special structures involving the heat and wave equations.

In Section 7 we consider well-posed linear systems with static nonlinear output feedback. These results are of small gain type, and they guarantee the well-posedness of the closed-loop system for certain Lipschitz constants of the nonlinearity, or the local well-posedness for bilinear feedbacks satisfying a certain estimate.

Sections 8 and 9 are devoted to examples. Using the nonlinear feedback theory from Section 7 we prove the global well-posedness of a system described by the Burgers equation and the local well-posedness of the Navier-Stokes equations on a bounded domain.

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2 Well-posedness in finite dimensions

Finite-dimensional linear control theory is mainly concerned with systems Σ described by equations of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \tag{2.1}$$

where u is the *input signal*, x is the *state trajectory*, y is the *output signal* and A, B, C, D are matrices of appropriate dimensions, dictated by the dimensions of the vectors $u(t), x(t)$ and $y(t)$. We denote by U, X and Y the (finite dimensional) *input space, state space* and *output space* of the system Σ, i.e., the spaces where $u(t), x(t)$ and $y(t)$ are. Usually such systems are considered to evolve over the time interval $\mathbb{R}_+ = [0, \infty)$, so that we have an initial state $x(0)$ and an input function u, while the functions x and y are uniquely determined by them. Applying the Laplace transformation (assuming that u has one), we obtain

$$\hat{y}(s) = C(sI - A)^{-1}x(0) + \mathbf{G}(s)\hat{u}(s), \tag{2.2}$$

where $\mathbf{G}(s) = C(sI-A)^{-1}B+D$ is the *transfer function* of the system. This is an $\mathcal{L}(U,Y)$-valued analytic function defined everywhere except on $\sigma(A)$, the spectrum of A. The matrix-valued function $\mathbf{G}$ is *rational*, meaning that each entry is a fraction of polynomials, and it is *proper*, meaning that it has a finite limit at infinity (which is D). We assume that this class of systems is familiar to the reader.

We have a lot of freedom in choosing the space of functions where u and y in (2.1) are. We could choose, for example, continuous functions, differentiable functions, various Sobolev spaces, functions of class L^p (where $1 \leqslant p \leqslant \infty$). However, in this survey, we mostly stick to the choice

$$u \in L^2_{\text{loc}}(\mathbb{R}; U), \qquad y \in L^2_{\text{loc}}(\mathbb{R}; Y).$$

For any interval J, we denote by $\mathbf{P}_J$ the operator of truncation of a vector-valued function v defined on a larger set than J, to J. The truncated function $\mathbf{P}_J v$ will sometimes (whenever this is convenient) be regarded as being defined on all $\mathbb{R}$, and having the value 0 ouside of J. An important feature of the system Σ that we want to emphasize is that on any finite time interval $[\tau, t] \subset \mathbb{R}$, if the initial state $x(\tau)$ and the corresponding segment of the input signal, $\mathbf{P}_{[\tau,t]}u$ are given, then we can solve (2.1) on $[\tau, t]$ (with a unique solution). We then have

$$\begin{bmatrix} x(t) \\ \mathbf{P}_{[t,\tau]}y \end{bmatrix} = \Sigma(t,\tau) \begin{bmatrix} x(\tau) \\ \mathbf{P}_{[t,\tau]}u \end{bmatrix}, \tag{2.3}$$

where $\Sigma(t,\tau)$ is a bounded linear operator from $X \times L^2([\tau,t]; U)$ to $X \times L^2([\tau,t]; Y)$, which can be naturally partinioned into four components. (It is an easy exercise to write these components explicitly.) If we restrict our solution (the functions u, x, y) to a subinterval of $[\tau, t]$, then (2.3) will hold also on this subinterval. This forces the components of $\Sigma(t,\tau)$ to obey certain algebraic rules, that we shall encounter later when discussing the abstract definition of a well-posed linear system.

In the sequel, we denote by $\mathcal{S}_h$ the bilateral right shift by h (where $h \in \mathbb{R}$) on $L^1_{\text{loc}}(\mathbb{R}, V)$, for any Banach space V. Thus, for $h > 0$, $\mathcal{S}_h u$ is the signal u delayed by the amount h, while for $h < 0$ it is u anticipated (brought earlier) by the amount $|h|$. The system Σ is called *linear* because the operators $\Sigma(t,\tau)$ are linear. The system Σ is called *well-posed* because the operators $\Sigma(t,\tau)$ are continuous (in the linear case discussed here, this is equivalent to them being bounded). The system Σ is called *time-invariant* because the operators $\Sigma(t,\tau)$ have the following property:

$$\begin{bmatrix} x(t) \\ \mathcal{S}_{-\tau}\mathbf{P}_{[t,\tau]}y \end{bmatrix} = \Sigma(t-\tau,0) \begin{bmatrix} x(\tau) \\ \mathcal{S}_{-\tau}\mathbf{P}_{[t,\tau]}u \end{bmatrix}, \tag{2.4}$$

which combined with (2.3) shows that essentially, $\Sigma(t,\tau)$ depends only on the time difference $t-\tau$. Σ should be called a *finite-dimensional linear time-invariant well-posed system* although you could not find this terminology in the systems and control literature, and for a good reason: all finite dimensional LTI systems are automatically well-posed, as it is easy to see from the solution formulas (that we did not write down). Thus, such systems are just called *finite-dimensional LTI systems*.

Now we turn our attention to finite-dimensional nonlinear time-invariant systems described by

$$\dot{x}(t) = f(x(t), u(t)), \tag{2.5}$$
$$y(t) = g(x(t), u(t)), \tag{2.6}$$

but for the moment we ignore (2.6) and just ask when can we solve the differential equation (2.5). It simplifies matters a lot if we still assume that $u(t) \in U$ and $x(t) \in X$, where U and X are finite-dimensional normed spaces.

Definition 2.1 *Let $f \in \mathcal{C}(X \times U, X)$, $\delta > 0$ and let $u : [0,\delta) \to U$ be measurable. A* solution *of (2.5) on $[0,\delta)$ is an absolutely continuous function $x : [0,\delta) \to X$ such that*

$$x(t) - x(0) = \int_0^t f(x(\sigma), u(\sigma))\mathrm{d}\sigma \quad \forall t \in [0,\delta).$$

The following theorem is a consequence of Theorem 36 (in Appendix C) in Sontag [61]. In the sequel, for any $c > 0$ we denote $\mathbf{B}_c = \{x \in X \mid \|x\| \leqslant c\}$.

Theorem 2.2 *Assume that $u : \mathbb{R}_+ \to U$ is measurable, $f \in \mathcal{C}(X \times U; X)$ and the following two conditions hold for every $a \in X$:*

(S1) *There exists a constant $c > 0$ and a locally integrable function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|f(x,u(t)) - f(y,u(t))\| \leqslant \alpha(t)\|x-y\|$$

for almost every $t \in \mathbb{R}_+$ and for all $x, y \in a + \mathbf{B}_c$.

(S2) *There exists a locally integrable function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|f(a,u(t))\| \leqslant \beta(t), \quad \textit{for almost every } t \in \mathbb{R}_+.$$

Then for every $x_0 \in X$ there exists $\delta > 0$ and a unique solution of (2.5) on $[0,\delta)$ satisfying $x(0) = x_0$. (The theorem remains valid if X is a Banach space.)

Thus, the existence of solutions can, in general, only be guaranteed locally, i.e., on some possibly short time interval. When the solution of (2.5) is not global, then it necessarily "blows up" in finite time, as the following corollary shows.

Corollary 2.3 *Suppose that u and f are as in Theorem 2.2 and for some $x_0 \in X$ and $\delta > 0$, $[0,\delta)$ is the maximal interval of existence of the solution of (2.5) with $x(0) = x_0$. Then for every $c > 0$ there exists $T \in [0,\delta)$ such that $x(T) \notin \mathbf{B}_c$.*

For the proof see, for instance, Jayawardhana *et al* [40].

If $\delta < \infty$ is as in Corollary 2.3, then it is called *the finite escape time* of the state trajectory x starting from x_0. Non-linear systems theory is plagued by finite escape times, and one possible technique that can help is to introduce an energy function that, for some reason, must stay bounded and thus prevents the "blow up" of the solutions. Energy considerations are useful in many other respects (stability analysis, control design). This is the next topic that we discuss.

Driven by physical examples, there has been much interest in systems that are passive, which means that they satisfy some sort of energy balance inequality. To define this concept we need a function $H \in \mathcal{C}^1(X;\mathbb{R}_+)$ called the *Hamiltonian* or *storage function*. This is often the physical energy stored in the system, but it does not have to be. We also need a real-valued function S called the *supply rate* defined on $U \times Y$ that is usually assumed to be continuous. The system is called *passive with respect to the storage function H and the supply rate S* if for any functions u, x and y that solve (2.5) and (2.6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x(t)) \leqslant S(u(t), y(t)). \tag{2.7}$$

H is called *proper* if $H(x) \to \infty$ when $\|x\| \to \infty$ or equivalently, for any constant $c > 0$ the set $\{x \in X \mid H(x) \leqslant c\}$ is compact. There is a huge literature on passive systems, of which we cite Willems [87] (who started it all) and van der Schaft [74].

Now we look at the linear time-invariant case. Let us assume that $U = Y$ and X are finite-dimensional inner product spaces. The time-invariant system Σ from (2.1) is called *impedance passive* if along solutions of (2.1),

$$\frac{\mathrm{d}}{\mathrm{d}t} \|x(t)\|^2 \leqslant 2\mathrm{Re}\, \langle u(t), y(t) \rangle. \tag{2.8}$$

This corresponds to taking in (2.7) $H(x) = \frac{1}{2}\|x\|^2$ and $S(u,y) = \mathrm{Re}\, \langle u, y\rangle$. It is easy to see that this is equivalent to the fact that, for every $x_0 \in X$ and $u_0 \in U$, we have

$$\mathrm{Re}\, \langle Ax_0 + Bu_0, x_0 \rangle \leqslant \mathrm{Re}\, \langle u_0, Cx_0 + Du_0 \rangle.$$

There is a very neat characterization of such systems due to Staffans [63,64], who did his investigations in the infinite-dimensional context.

Theorem 2.4 *The system Σ from* (2.1) *is impedance passive in the sense of* (2.8) *if and only if the matrix*

$$T = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}$$

is dissipative (i.e., $T + T^* \leqslant 0$).

It is now obvious that if Σ is impedance-passive, then A is dissipative. This is equivalent to the fact that e^{At} is a contraction for every $t \geqslant 0$. When $D + D^*$ is invertible, then impedance passivity is equivalent to $D + D^* > 0$ and

$$A + A^* + (C^* - B)(D + D^*)^{-1}(C - B^*) \leqslant 0.$$

In the case that $D = 0$, impedance passivity is equivalent to $C = B^*$ and A being dissipative. It is not difficult to check that if $\mathbf{G}$ is the transfer function of an impedance passive system, then $\mathbf{G}$ is *positive*, meaning that

$$\mathbf{G}(s) + \mathbf{G}(s)^* \geqslant 0 \qquad \forall\, s \in \mathbb{C}_0, \tag{2.9}$$

where we have used the notation

$$\mathbb{C}_\alpha = \{s \in \mathbb{C} \mid \mathrm{Re}\, s > \alpha\}.$$

It is easy to derive versions of the above statements that involve weighting operators on the spaces U and Y - we shall not waste time on this.

Now let us drop the assumption that $U = Y$. The time-invariant system Σ from (2.1) is called *scattering passive* if along solutions of (2.1),

$$\frac{\mathrm{d}}{\mathrm{d}t} \|x(t)\|^2 \leqslant \|u(t)\|^2 - \|y(t)\|^2. \tag{2.10}$$

This corresponds to taking in (2.7) $H(x) = \frac{1}{2}\|x\|^2$ and $S(u,y) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\|y\|^2$. It is easy to see that this is equivalent to the fact that, for $x_0 \in X$ and $u_0 \in U$,

$$2\mathrm{Re}\, \langle Ax_0 + Bu_0, x_0 \rangle \leqslant \|u_0\|^2 - \|Cx_0 + Du_0\|^2.$$

It is also easy to see that Σ is scattering passive if and only if the operators $\Sigma(t,\tau)$ from (2.3) are contractions.

Theorem 2.5 *The system Σ from* (2.1) *is scattering passive in the sense of* (2.10) *if and only if*

$$\begin{bmatrix} A + A^* & B & C^* \\ B^* & -I & D^* \\ C & D & -I \end{bmatrix} \leqslant 0.$$

This result has been obtained by adapting a related result from Apkarian, Gahinet and Becker, [6], who have worked in the finite-dimensional parameter-varying context. The last theorem can also be derived from Theorem 2.4, by applying it to the system $\tilde{\Sigma}$ with inputs u_1, u_2 and outputs y_1, y_2 defined by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu_1(t), \\ y_1(t) = \frac{1}{2}u_1(t), \\ y_2(t) = -Cx(t) - Du_1(t) + \frac{1}{2}u_2(t). \end{cases}$$

It is not difficult to check that if Σ is scattering passive, then the matrix A is dissipative (so that $\mathbb{C}_0 \subset \rho(A)$) and the transfer function $\mathbf{G}$ of the system is a *Schur function*, i.e., it satisfies $\|\mathbf{G}(s)\| \leqslant 1$ for all $s \in \mathbb{C}_0$.

It is useful to note that most scattering passive systems can be obtained from impedance scattering ones via the *external Cayley transformation* (sometimes called the diagonal transformation), which redefines the input and the output as follows: If e and f are the input and output signals of an impedance passive system $\Sigma_{\rm imp}$, then the input and output signals of the corresponding scattering passive system $\Sigma_{\rm sca}$ are

$$u = \tfrac{1}{\sqrt{2}}(e+f), \qquad y = \tfrac{1}{\sqrt{2}}(e-f). \tag{2.11}$$

The inverse transformation is given by the same formulas, only with the places of u, y and e, f reversed, as is easy to see. This transformation has been employed in many works, see for example Staffans [63,64,69], and Weiss [80]. The external Cayley transformation can be understood also as an output feedback transformation (combined with a feed-forward term and a rescaling), as Figure 1 (approximately reproduced from [80]) shows. It is easy to see from this figure that the relation between the transfer functions of $\Sigma_{\rm imp}$ and $\Sigma_{\rm sca}$ is

$$\mathbf{G}_{\rm sca} = (I - \mathbf{G}_{\rm imp})(I + \mathbf{G}_{\rm imp})^{-1}.$$

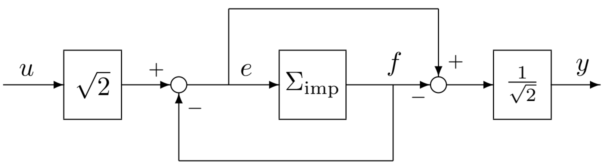

Fig. 1. The system $\Sigma_{\rm sca}$ with input u and output y, as obtained from the system $\Sigma_{\rm imp}$ (with input e and output f) via the external Cayley transformation (2.11).

In [63,64] the precise relationship between $\Sigma_{\rm imp}$ and $\Sigma_{\rm sca}$ has been determined (in the infinite-dimensional context), as stated in the following proposition.

Proposition 2.6 *Suppose that* $A_{\rm imp}, B_{\rm imp}, C_{\rm imp}, D_{\rm imp}$ *determine via* (2.1) *an impedance passive system* $\Sigma_{\rm imp}$. *Then the matrix*

$$E_{\rm imp} := \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} I & 0 \\ C_{\rm imp} & I + D_{\rm imp} \end{bmatrix} \tag{2.12}$$

is invertible. Define the system $\Sigma_{\rm sca}$ *via its matrices* $A_{\rm sca}, B_{\rm sca}, C_{\rm sca}, D_{\rm sca}$ *by*

$$\begin{bmatrix} A_{\rm sca} & B_{\rm sca} \\ C_{\rm sca} & D_{\rm sca} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} A_{\rm imp} & B_{\rm imp} \\ 0 & \sqrt{2} I \end{bmatrix} E_{\rm imp}^{-1}. \tag{2.13}$$

Then $\Sigma_{\rm sca}$ *is a scattering passive system. We have* $E_{\rm imp}^{-1} = E_{\rm sca}$, *where*

$$E_{\rm sca} := \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} I & 0 \\ C_{\rm sca} & I + D_{\rm sca} \end{bmatrix}, \tag{2.14}$$

and $\Sigma_{\rm imp}$ *can be recovered from* $\Sigma_{\rm sca}$ *via*

$$\begin{bmatrix} A_{\rm imp} & B_{\rm imp} \\ C_{\rm imp} & D_{\rm imp} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} A_{\rm sca} & B_{\rm sca} \\ 0 & \sqrt{2} I \end{bmatrix} E_{\rm sca}^{-1}. \tag{2.15}$$

Notice that $\Sigma_{\rm sca}$ is obtained from $\Sigma_{\rm imp}$ by the same formulas by which $\Sigma_{\rm imp}$ is obtained from $\Sigma_{\rm sca}$. However, there is a hidden asymmetry here: the external Cayley transformation will not yield every possible scattering passive system. It follows from the above proposition that the range of the external Cayley transformation is the set of those scattering passive systems for which $I + D_{\rm sca}$ is invertible.

3 Well-posed LTI systems on Hilbert spaces

For infinite-dimensional LTI systems, we think that it is not a good idea to start from equations of the type (2.1), or any other types of differential and algebraic equations that describe the system locally in time (i.e., at one instant). This is because we encounter differential, trace and other unbounded operators, and it is difficult to build a general and clear definition for a well-posed system using these. Indeed, we would get bogged down endlessly in choosing the right domains and trying to define what we mean by solutions of certain equations. Starting with the work of Salamon [56,57] the accepted approach is to start from the global (or "integral") operators $\Sigma(t,\tau)$ that appear in (2.3), which are bounded. This is similar to the theory of *operator semigroups* (we use this name for what is also known as strongly continuous semigroups, or C_0 semigroups, and we assume that the reader is familiar with them). Indeed, in the theory of operator semigroups, the definition concerns the family of (global) operators in the semigroup, which are bounded, and the (usually unbounded) generator appears later in the theory. Similarly, we start with the family of bounded operators $\Sigma(t,\tau)$ and the (usually unbounded) operators in (2.1) will appear later. There are many competing and equivalent definitions for a well-posed LTI system, see for instance Salamon [56], Weiss [77,79], Staffans [67,68,65]. We shall use the definition from [79] (which is often employed).

The idea of the definition is that the system is fully described by the operators $\Sigma(t-\tau, 0)$ appearing in (2.4). For convenience we denote $\Sigma_\tau = \Sigma(\tau, 0)$. We partition these operators as follows:

$$\Sigma_\tau = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix} \qquad \forall\, \tau \geqslant 0. \tag{3.1}$$

The definition will list the requirements that have to be imposed on each of the four component operator families, so that the concept corresponds to what we expect based on our intuition and experience. Of course, finite-dimensional LTI systems (as discussed in the first section) must be a particular case of well-posed LTI system in the Hilbert space context, as defined below. Well-posed LTI systems are most often called *well-posed linear systems.*

Notation. Let W be a Hilbert space. For any interval J, we regard $L^2_{\rm loc}(J;W)$ as a subspace of $L^2_{\rm loc}(\mathbb{R};W)$ (by extending functions defined on J with the value 0 outside J). Recall the truncation operators $\mathbf{P}_J$ and the bilateral right shift operators $\mathcal{S}_\tau$ introduced in Section 2. For any $u, v \in L^2_{\rm loc}([0,\infty);W)$ and any $\tau \geqslant 0$, the *τ-concatenation* of u and v is the function defined by

$$u \underset{\tau}{\diamondsuit} v = \mathbf{P}_{[0,\tau]} u + \mathcal{S}_\tau v.$$

Thus, $(u \underset{\tau}{\diamondsuit} v)(t) = u(t)$ for $t \in [0,\tau)$, while $(u \underset{\tau}{\diamondsuit} v)(t) = v(t-\tau)$ for $t \geqslant \tau$. If $\mathbb{T}$ is an operator semigroup, we denote its growth bound by $\omega_0(\mathbb{T})$.

Definition 3.1 *Let U, X and Y be Hilbert spaces. A* well-posed linear system *is a family of operators $\Sigma = (\Sigma_t)_{t\geqslant 0}$ partitioned as in* (3.1)*, where*

(i) *$\mathbb{T} = (\mathbb{T}_t)_{t\geqslant 0}$ is an operator semigroup on X,*

(ii) *$\Phi = (\Phi_t)_{t\geqslant 0}$ is a family of bounded linear operators from $L^2([0,\infty);U)$ to X such that*

$$\Phi_{\tau+t}(u \underset{\tau}{\diamondsuit} v) = \mathbb{T}_t \Phi_\tau u + \Phi_t v, \tag{3.2}$$

for every $u, v \in L^2([0,\infty);U)$ and all $\tau, t \geqslant 0$,

(iii) *$\Psi = (\Psi_t)_{t\geqslant 0}$ is a family of bounded linear operators from X to $L^2([0,\infty);Y)$ such that*

$$\Psi_{\tau+t} x_0 = \Psi_\tau x_0 \underset{\tau}{\diamondsuit} \Psi_t \mathbb{T}_\tau x_0, \tag{3.3}$$

for every $x_0 \in X$ and all $\tau, t \geqslant 0$, and $\Psi_0 = 0$,

(iv) *$\mathbb{F} = (\mathbb{F}_t)_{t\geqslant 0}$ is a family of bounded linear operators from $L^2([0,\infty);U)$ to $L^2([0,\infty);Y)$ such that*

$$\mathbb{F}_{\tau+t}(u \underset{\tau}{\diamondsuit} v) = \mathbb{F}_\tau u \underset{\tau}{\diamondsuit} (\Psi_t \Phi_\tau u + \mathbb{F}_t v), \tag{3.4}$$

for every $u, v \in L^2([0,\infty);U)$ and all $\tau, t \geqslant 0$, and $\mathbb{F}_0 = 0$.

We call U the input space*, X the* state space *and Y the* output space *of Σ. The operators Φ_τ are called* input maps*, the operators Ψ_τ are called* output maps*, and the operators $\mathbb{F}_\tau$ are called* input-output maps*.*

It is often convenient to denote $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ in place of arranging these families in a 2×2 matrix as in (3.1).

It follows from (3.2) with $t = 0$ and $v = 0$ that Φ is *causal*, the state does not depend on the future input: $\Phi_\tau \mathbf{P}_\tau = \Phi_\tau$ for all $\tau \geqslant 0$, in particular $\Phi_0 = 0$. It follows from this and the definitions that for all $\tau, t \geqslant 0$,

$$\Phi_{\tau+t}\mathbf{P}_{[0,\tau]} = \mathbb{T}_t \Phi_\tau, \qquad \mathbf{P}_{[0,\tau]}\Psi_{\tau+t} = \Psi_\tau,$$

$$\mathbf{P}_{[0,\tau]}\mathbb{F}_{\tau+t}\mathbf{P}_{[0,\tau]} = \mathbf{P}_{[0,\tau]}\mathbb{F}_{\tau+t} = \mathbb{F}_\tau,$$

and hence $\mathbf{P}_{[0,\tau]}\mathbb{F}_{\tau+t}\mathbf{P}_{[\tau,\tau+t]} = 0$. The last identity says that $\mathbb{F}$ is *causal* (i.e., the past output does not depend on the future input).

Example 3.2 We give an extremely simple but important example of an infinite-dimensional well-posed system, borrowed from [79]. We model a delay line as a well-posed linear system. Let $X = L^2[-h,0]$, where $h > 0$, and let $\mathbb{T}$ be the left shift semigroup on X with zero entering from the right, i.e., for any $\tau \geqslant 0$ and $\zeta \in [-h,0]$,

$$(\mathbb{T}_\tau x)(\zeta) = \begin{cases} x(\zeta+\tau), & \text{for } \zeta+\tau \leqslant 0, \\ 0, & \text{for } \zeta+\tau > 0. \end{cases}$$

Let $U = \mathbb{C}$ and for any $\tau \geqslant 0$ and $\zeta \in [-h,0]$ define

$$(\Phi_\tau u)(\zeta) = \begin{cases} u(\zeta+\tau), & \text{for } \zeta+\tau \geqslant 0, \\ 0, & \text{for } \zeta+\tau < 0. \end{cases}$$

Let $Y = \mathbb{C}$ and for any $\tau \geqslant 0$ and $t \in [0,\tau)$ define

$$(\Psi_\tau x_0)(t) = \begin{cases} x(t-h), & \text{for } t-h \leqslant 0, \\ 0, & \text{for } t-h > 0. \end{cases}$$

For $t \geqslant \tau$ we put $(\Psi_\tau x)(t) = 0$. Finally, let for any $\tau \geqslant 0$ and $t \in [0,\tau)$

$$(\mathbb{F}_\tau u)(t) = \begin{cases} u(t-h), & \text{for } t-h \geqslant 0, \\ 0, & \text{for } t-h < 0. \end{cases}$$

For $t \geqslant \tau$ we put $(\mathbb{F}_\tau u)(t) = 0$. Then $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is a well-posed linear system. It is clear from the formula of $\mathbb{F}$ that this is indeed a delay line of size h.

For the remainder of this section we use the assumptions of Definition 3.1. We denote the generator of $\mathbb{T}$ by A. The space X_1 is defined as $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, where $\beta \in \rho(A)$, and X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1} z\|$. The choice of β is not important, since different choices lead to equivalent norms on X_1 and on X_{-1}. In fact, the norm $\|\cdot\|_1$ is equivalent to the graph norm on $\mathcal{D}(A)$ and X_{-1} may be regarded as the dual of $\mathcal{D}(A^*)$ (with its graph norm) with respect to the pivot space X. Thus,

$$X_1 \subset X \subset X_{-1} \tag{3.5}$$

densely and with continuous embeddings. The semigroup $\mathbb{T}$ can be extended to X_{-1}, and then its generator is an extension of A, whose domain is X. We use the same notation for all these extensions as for the original operators. The extended semigroup is isomorphic to the original one via the isomorphism $(\beta I - A) \in \mathcal{L}(X, X_{-1})$.

We denote the corresponding spaces that we get by replacing A with A^* by X_1^d and X_{-1}^d, i.e., X_1^d is $\mathcal{D}(A^*)$ with the norm $\|z\|_1^d = \|(\beta I - A^*)z\|$, and X_{-1}^d is the completion of X with respect to the norm $\|z\|_{-1}^d = \|(\beta I - A^*)^{-1}z\|$. The scalar product of X has continuous extensions to $X_1 \times X_{-1}^d$ and to $X_1^d \times X_{-1}$, so that X_{-1}^d (respectively X_{-1}) may be regarded as the dual of X_1 (respectively of X_1^d) with respect to the pivot space X. More details about these spaces and other related ones, such as X_{-2}, can be found in Engel and Nagel [23], Staffans [65] and [73].

For the remainder of this section we recall some less immediate consequences of Definition 3.1, following [67,68,73,79], mostly without proof.

A nontrivial consequence of assumptions (i) and (ii) in the definition is that there exists a unique $B \in \mathcal{L}(U, X_{-1})$, called the *control operator* of Σ, such that

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) \,\mathrm{d}\sigma \quad \forall\, t \geqslant 0. \tag{3.6}$$

Notice that in the above formula, $\mathbb{T}$ acts on X_{-1} and the integration is carried out in X_{-1}. $\Phi_t u$ depends continuously on t. The operator B can be found by

$$B\mathrm{v} = \lim_{\tau \to 0} \frac{1}{\tau} \Phi_\tau(\chi \cdot \mathrm{v}) \qquad \forall\, \mathrm{v} \in U, \tag{3.7}$$

where χ denotes the characteristic function of $[0, \infty)$.

Remark 3.3 Let U, X be Hilbert spaces and let $\mathbb{T}$ be an operator semigroup on X. An operator $B \in \mathcal{L}(U, X_{-1})$ is called an *admissible control operator* for $\mathbb{T}$ if for some $t > 0$, the integral in (3.6) is in X, for any $u \in L^2([0, \infty); U)$. If this is the case, then $\Phi_t \in \mathcal{L}(L^2([0, \infty); U), X)$ for all $t \geqslant 0$. If $B \in \mathcal{L}(U, X)$, then obviously it is admissible. Such control operators are called *bounded*, and the others are called *unbounded*. Clearly, if B is the control operator of a well-posed system, then it is an admissible control operator for the operator semigroup of the system.

We do not want to spend much space in this survey on the concept of an admissible control operator, and for more details we refer to the excellent survey by Jacob and Partington [35], as well as to Jacob, Partington and Pott [36], Staffans [65], Tucsnak and Weiss [73] and Weiss [75]. We mention here only three important results:

(1) Suppose that $\mathbb{T}$ is exponentially stable, i.e., $\omega_0(\mathbb{T}) < 0$. Then B is an admissible control operator for $\mathbb{T}$ if and only if the equation

$$A\Pi z + \Pi A^* z = -BB^* z \qquad \forall\, z \in \mathcal{D}(A^*)$$

(called a Lyapunov equation) has a solution $\Pi \in \mathcal{L}(X)$ with $\Pi \geqslant 0$. (This solution is then unique, and is called the controllability Gramian of A and B.)

(2) Suppose that $\mathbb{T}$ is left-invertible. Then B is an admissible control operator for $\mathbb{T}$ if and only if, for some $\omega > \omega_0(\mathbb{T})$,

$$\sup_{\operatorname{Re} s = \omega} \|(sI - A)^{-1} B\|_{\mathcal{L}(U,X)} < \infty.$$

(3) If B is an admissible control operator for $\mathbb{T}$ then for every $\omega > \omega_0(\mathbb{T})$,

$$\sup_{\operatorname{Re} s > \omega} (\operatorname{Re} s) \|(sI - A)^{-1} B\|^2_{\mathcal{L}(U,X)} < \infty. \tag{3.8}$$

This is the easy part. The strong result is that the converse holds under additional assumptions: Suppose that U is finite-dimensional and $\mathbb{T}$ is a contraction semigroup, or it is normal. Then (3.8) (for one $\omega \in \mathbb{R}$) implies that B is admissible for $\mathbb{T}$. For related results and extensions see, e.g., Haak and Kunstmann [30], and the counterexamples in Jacob and Zwart [37] and Zwart, Jacob and Staffans [90].

Now we turn our attention to the output maps of the well-posed system Σ from Definition 3.1. It follows from the identity $\mathbf{P}_{[0,\tau]} \Psi_{\tau+t} = \Psi_\tau$ (for $\tau, t \geqslant 0$) that there exists a unique operator $\Psi_\infty : X \to L^2_{\mathrm{loc}}([0, \infty); Y)$ such that $\mathbf{P}_{[0,\tau]} \Psi_\infty = \Psi_\tau$ for all $\tau \geqslant 0$. Ψ_∞ is called the *extended output map* of Σ, and it satisfies

$$\Psi_\infty x_0 = \Psi_\infty x_0 \mathop{\diamondsuit}_{\tau} \Psi_\infty \mathbb{T}_\tau x_0, \tag{3.9}$$

for every $x_0 \in X$ and all $\tau \geqslant 0$. It can be shown (using assumptions (i) and (iii) in the definition) that there exists a unique $C \in \mathcal{L}(X_1, Y)$, called the *observation operator* of Σ, such that for every $x_0 \in \mathcal{D}(A)$ and all $t \geqslant 0$,

$$(\Psi_\infty x_0)(t) = C\mathbb{T}_t x_0. \tag{3.10}$$

This determines Ψ_∞, since $\mathcal{D}(A)$ is dense in X.

Remark 3.4 An operator $C \in \mathcal{L}(X_1, Y)$ is called an *admissible observation operator* for $\mathbb{T}$ if the estimate

$$\int_0^\tau \|C\mathbb{T}_t x_0\|^2 \,\mathrm{d}t \leqslant k\|x_0\|^2$$

holds for some (hence, for every) $\tau > 0$ and for every $x_0 \in \mathcal{D}(A)$. The constant $k \geqslant 0$ may depend on τ. If $C \in \mathcal{L}(X, Y)$ then obviously it is admissible. Such observation operators are called *bounded*, while the others are called *unbounded*. It is clear that if C is the observation operator of a well-posed linear system Σ, then C is admissible for the semigroup $\mathbb{T}$ of Σ. For further details about admissible observation operators we refer to Weiss [76] as well as all the references in Remark 3.3. The connection with admissible control operators is the following duality: $C \in \mathcal{L}(X_1, Y)$ *is an admissible observation operator for* $\mathbb{T}$ *if and only if* $C^* \in \mathcal{L}(Y, X_{-1}^d)$ *is an admissible control operator for* $\mathbb{T}^*$. In particular, the dual version of the estimate (3.8) is the following: if C is an admissible observation operator for $\mathbb{T}$ then for every $\omega > \omega_0(\mathbb{T})$,

$$\sup_{\operatorname{Re} s > \omega} (\operatorname{Re} s) \|C(sI - A)^{-1}\|^2_{\mathcal{L}(X,Y)} < \infty . \tag{3.11}$$

Now we turn our attention to the input-output maps of the well-posed system Σ. It follows from the identity $\mathbf{P}_{[0,\tau]} \mathbb{F}_{\tau+t} = \mathbb{F}_\tau$ that there exists a unique linear operator $\mathbb{F}_\infty : L^2_{\text{loc}}([0,\infty); U) \to L^2_{\text{loc}}([0,\infty); Y)$ such that $\mathbf{P}_{[0,\tau]} \mathbb{F}_\infty = \mathbb{F}_\tau$ for all $\tau \geqslant 0$. This $\mathbb{F}_\infty$ is called the *extended input-output map* of Σ. We have

$$\mathbb{F}_\infty(u \underset{\tau}{\diamondsuit} v) = \mathbb{F}_\infty u \underset{\tau}{\diamondsuit} (\Psi_\infty \Phi_\tau u + \mathbb{F}_\infty v), \tag{3.12}$$

for every $u, v \in L^2_{\text{loc}}([0,\infty); U)$ and all $\tau \geqslant 0$. Taking $u = 0$ in (3.12) we get that

$$\mathbb{F}_\infty \mathcal{S}_\tau = \mathcal{S}_\tau \mathbb{F}_\infty , \tag{3.13}$$

for every $\tau \geqslant 0$. This property means that $\mathbb{F}_\infty$ is *shift-invariant* or *time-invariant*.

Notation. For any Hilbert space W, any interval J and any $\omega \in \mathbb{R}$ we put

$$L^2_\omega(J; W) = e_\omega L^2(J; W),$$

where $(e_\omega v)(t) = e^{\omega t} v(t)$, with the norm $\|e_\omega v\|_{L^2_\omega} = \|v\|_{L^2}$. We denote by $\mathbb{C}_\omega = \{s \in \mathbb{C} \mid \operatorname{Re} s > \omega\}$.

It can be shown that for every $\omega > \omega_0(\mathbb{T})$, Ψ_∞ is bounded from X to $L^2_\omega([0,\infty); Y)$. For each $x_0 \in X$, the Laplace integral of $\Psi_\infty x_0$ converges absolutely for $\operatorname{Re} s > \omega_0(\mathbb{T})$, and for such values of s the Laplace transform is given by

$$(\widehat{\Psi_\infty x_0})(s) = C(sI - A)^{-1} x_0 . \tag{3.14}$$

We denote by $\gamma_{\mathbb{F}}$ the infimum of those $\omega \in \mathbb{R}$ for which $\mathbb{F}_\infty$ is bounded from $L^2_\omega([0,\infty); U)$ to $L^2_\omega([0,\infty); Y)$. This number $\gamma_{\mathbb{F}} \in [-\infty, \infty)$ is called the *growth bound* of $\mathbb{F}_\infty$. It can be shown that $\gamma_{\mathbb{F}} \leqslant \omega_0(\mathbb{T})$. We can represent $\mathbb{F}_\infty$ via the *transfer function* $\mathbf{G}$ of Σ, which is a bounded analytic $\mathcal{L}(U, Y)$-valued function on $\mathbb{C}_\omega$ for every $\omega > \gamma_{\mathbb{F}}$. If $u \in L^2_\omega([0,\infty); U)$ with $\omega > \gamma_{\mathbb{F}}$ then the Laplace integral of $\mathbb{F}_\infty u$ converges absolutely for $\operatorname{Re} s > \gamma_{\mathbb{F}}$ and

$$(\widehat{\mathbb{F}_\infty u})(s) = \mathbf{G}(s) \hat{u}(s), \qquad \operatorname{Re} s > \omega_{\mathbb{F}} . \tag{3.15}$$

The transfer function $\mathbf{G}$ satisfies

$$\begin{aligned} \mathbf{G}(s) - \mathbf{G}(\beta) &= (\beta - s) C(\beta I - A)^{-1} (sI - A)^{-1} B \\ &= C \left[(sI - A)^{-1} - (\beta I - A)^{-1} \right] B, \end{aligned} \tag{3.16}$$

for all $s, \beta \in \mathbb{C}_{\omega_0(\mathbb{T})}$ (equivalently, $\mathbf{G}'(s) = -C(sI - A)^{-2} B$). This shows that $\mathbf{G}$ is determined by A, B and C up to an additive constant operator.

The growth bound $\gamma_{\mathbb{F}}$ is the infimum of all those $\omega \in \mathbb{R}$ for which $\mathbf{G}$ has a bounded analytic extension to $\mathbb{C}_\omega$. It follows from (3.15) and the Paley-Wiener theorem that for $\omega > \gamma_{\mathbb{F}}$, the norm of $\mathbb{F}_\infty$ from L^2_ω to L^2_ω is the supremum of $\|\mathbf{G}(s)\|$ over all $s \in \mathbb{C}_\omega$. By the maximum modulus theorem, denoting $\|\mathbb{F}_\infty\|_\omega = \|\mathbb{F}_\infty\|_{\mathcal{L}(L^2_\omega)}$,

$$\|\mathbb{F}_\infty\|_\omega = \sup_{\operatorname{Re} s = \omega} \|\mathbf{G}(s)\| . \tag{3.17}$$

An analytic function defined on a domain that contains some right half-plane is called *proper* if it is bounded on some right half-plane (such as $\mathbf{G}$ above). This concept is the natural generalization of the well-known concept of properness for rational functions, that has been recalled in the text after (2.2).

There are transformations which lead from one well-posed system to another: duality, time-inversion, flow-inversion and time-flow inversion. Here we briefly recall duality, and we refer to [50,65,68] for the other (more challenging) transformations.

Notation. Let W be a Hilbert space. For every $u \in L^2_{\text{loc}}([0,\infty); W)$ and all $\tau \geqslant 0$, we define the *time-inversion operator* on $[0, \tau]$ as follows:

$$(\mathbf{Я}_\tau u)(t) = \begin{cases} u(\tau - t) & \text{for } \ t \in [0, \tau], \\ 0 & \text{for } \ t > \tau. \end{cases}$$

Theorem 3.5 *Let* $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ *be a well-posed linear system with input space* U*, state space* X *and output space* Y*. Define* Σ^d_τ *(for all* $\tau \geqslant 0$*) by*

$$\Sigma^d_\tau = \begin{bmatrix} \mathbb{T}^d_\tau & \Phi^d_\tau \\ \Psi^d_\tau & \mathbb{F}^d_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathbf{Я}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{T}^*_\tau & \Psi^*_\tau \\ \Phi^*_\tau & \mathbb{F}^*_\tau \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{Я}_\tau \end{bmatrix} . \tag{3.18}$$

Then $\Sigma^d = (\mathbb{T}^d, \Phi^d, \Psi^d, \mathbb{F}^d)$ *is a well-posed linear system with input space* Y*, state space* X *and output space* U*. If* A*,* B *and* C *are the semigroup generator, control operator and observation operator of* Σ*, then the corresponding operators for* Σ^d *are* A^**,* C^* *and* B^**. The transfer functions are related by*

$$\mathbf{G}^d(s) = \mathbf{G}^*(\bar{s}), \qquad \operatorname{Re} s > \omega_0(\mathbb{T}).$$

Both types of growth bounds are equal: $\omega_0(\mathbb{T}) = \omega_0(\mathbb{T}^d)$ *and* $\gamma_{\mathbb{F}} = \gamma_{\mathbb{F}^d}$*.*

The system Σ^d introduced above is called the *dual system* corresponding to Σ. It is easy to verify (from (3.18)) that applying the duality transformation twice, we get back the original system: $(\Sigma^d)^d = \Sigma$.

4 System nodes and solutions of system equations

Well-posed linear systems have various generalizations within the LTI context. One obvious one is to replace Hilbert spaces with Banach spaces and L^2 with L^p - this is one of the issues that we shall ignore in this paper, but we refer to relevant parts of [10,12,30,32,65,75,76,79] (this is an incomplete list).

A more interesting generalization is the concept of a *resolvent linear system*, due to Mark Opmeer [54,55], where the relations between input, state and output are formulated entirely in a sort of Laplace transformed domain. This allows "much less well-posed" systems to be included. The system is determined by four operator families, like well-posed systems, but these are analytic operator-valued functions which, in the case of a well-posed linear system and using the notation of Section 3, would have the interpretation of $(sI-A)^{-1}$, $(sI-A)^{-1}B$, $C(sI-A)^{-1}$ and $\mathbf{G}(s)$. The *integrated resolvent linear systems*, also introduced in [54], are a subclass that comes closer to (and still contains) well-posed systems. Other classes of systems that contain the well-posed ones (and are contained in resolvent linear systems) are the systems with n-admissible control and observation operators discussed in Latushkin *et al* [44], the systems that are *strictly proper with an integrator*, introduced in Weiss and Zhao [85], and the system nodes, presented below.

For the study of well-posed linear systems, the most useful generalization of the concept seems to be the concept of a system node. The reason for this is that system nodes look very much like well-posed systems decribed by equations local in time, but with most of the well-posedness assumptions deleted. Thus, when given a system of differential and algebraic equations that we "suspect" to be well-posed, we can, as a first step, verify that it is a system node after introducing the correct spaces and operators, and often this is relatively easy. After this is done, we know that the equations of the system have classical solutions for a significant space of initial conditions and input functions. Now, if we want to check the well-posedness of the system, sometimes we can do this quickly by using various theorems formulated in the abstract language of system nodes. For example, if we somehow know that the system node is scattering passive, then its well-posedness follows.

The idea of system nodes goes back to Y.L. Smuljan in 1986, using a different terminology. The concept as used today was formulated while writing Malinen *et al* [51] and we refer to that paper for the relation with earlier concepts such as operator colligations. System nodes have been used in many works, starting with Staffans [62], and good introductions are in Staffans [65] and Staffans and Weiss [69]. The definition given below is less elegant, but very short and easy to understand. Regarding classical and generalized solutions, our exposition follows [69].

Definition 4.1 *Suppose that A is the generator of a strongly continuous semigroup $\mathbb{T}$ on the Hilbert space X. In the sequel we use the spaces X_1 and X_{-1} and the extensions of A and $\mathbb{T}$, as introduced around* (3.5).

Let U and Y be Hilbert spaces, $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$. Let the function $\mathbf{G} : \mathbb{C}_{\omega_0(\mathbb{T})} \to \mathcal{L}(U, Y)$ be such that, for every $s, \beta \in \mathbb{C}_{\omega_0(\mathbb{T})}$,

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C[(sI-A)^{-1} - (\beta I - A)^{-1}]B. \quad (4.1)$$

Then $\Sigma_{node} = (A, B, C, \mathbf{G})$ is called a system node *on (U, X, Y). We call U, X, Y the input space, state space and output space of Σ_{node} respectively. A is the semigroup generator of Σ_{node}, B is its control operator, C is its observation operator, $\mathbf{G}$ is its transfer function and (A, B, C) is its generating triple.*

Notice that $\mathbf{G}$ is analytic and it satisfies

$$\mathbf{G}'(s) = -C(sI-A)^{-2}B \qquad \forall\, s \in \mathbb{C}_{\omega_0(\mathbb{T})}. \quad (4.2)$$

For any triple (A, B, C) as above we can find infinitely many functions $\mathbf{G}$ satisfying (4.1) (or, equivalently, (4.2)) and any two such functions differ by a constant.

The *combined observation/feedthrough operator* of Σ_{node} is defined by

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} = C[x - (\beta I - A)^{-1}Bu] + \mathbf{G}(\beta)u, \quad (4.3)$$

with domain

$$\mathcal{D}(C\&D) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in X \times U \;\middle|\; Ax + Bu \in X \right\}.$$

Note that the operator $C\&D$ is independent of the choice of $\beta \in \mathbb{C}_{\omega_0(\mathbb{T})}$ -this can be verified using (4.1). We have the following relation between $C\&D$ and $\mathbf{G}$:

$$\mathbf{G}(s) = C\&D \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix} \qquad \forall\, s \in \mathbb{C}_{\omega_0(\mathbb{T})}. \quad (4.4)$$

It may be that $\mathbf{G}$ has analytic extensions to half-planes $\mathbb{C}_\omega$ with $\omega < \omega_0(\mathbb{T})$. *We do not distinguish between an analytic function defined on a right half-plane and an* *analytic extension to a larger right half-plane.* This convention is important since (when $\rho(A)$ is not connected) we may get points $s \in \rho(A)$ where (4.4) is not true, see Curtain and Zwart [22, Example 4.3.8]. Thus, to avoid mistakes, when we are at a point s to the left of $\omega_0(\mathbb{T})$, we define $\mathbf{G}(s)$ by analytically extending $\mathbf{G}$, starting from the domain $\mathbb{C}_{\omega_0(\mathbb{T})}$ (if such an extension exists).

The natural norm on $\mathcal{D}(C\&D)$ is

$$\left\| \begin{bmatrix} x \\ u \end{bmatrix} \right\|^2_{\mathcal{D}(C\&D)} = \|x\|^2_X + \|u\|^2_U + \|Ax + Bu\|^2_X . \quad (4.5)$$

With this norm, $\mathcal{D}(C\&D)$ is a Hilbert space and

$$C\&D \in \mathcal{L}(\mathcal{D}(C\&D), Y). \quad (4.6)$$

The system node $\Sigma_{node} = (A, B, C, \mathbf{G})$ can also be determined by its *system operator*

$$S = \begin{bmatrix} A & B \\ \multicolumn{2}{c}{C\&D} \end{bmatrix}, \qquad \mathcal{D}(S) = \mathcal{D}(C\&D), \quad (4.7)$$

which is a densely defined and closed operator from $X \times U$ to $X \times Y$. In several papers, such as [69], S is taken as the starting point when defining a system node, and the operators A, B, C and the transfer function $\mathbf{G}$ are derived from S.

Define the space

$$Z = D(A) + (\beta I - A)^{-1} BU, \quad (4.8)$$

which is independent of $\beta \in \rho(A)$ and is a Hilbert space with the norm

$$\|z\|^2_Z = \inf \left\{ \|x\|^2_1 + \|v\|^2 \;\middle|\; \begin{matrix} x \in X_1,\ v \in U \\ z = x + (\beta I - A)^{-1} B v \end{matrix} \right\}.$$

Note that if $\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in \mathcal{D}(S)$, then $x \in Z$ and we have $\|x\|_Z \leqslant m \| \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \|_{\mathcal{D}(C\&D)}$, for some $m > 0$ independent of x and v. The system node is called *compatible* if C has a continuous extension to an operator $\overline{C} \in \mathcal{L}(Z, Y)$. In this case, we may define the operator $D \in \mathcal{L}(U, Y)$ by $D = \mathbf{G}(\beta) - \overline{C}(\beta I - A)^{-1} B$ and it follows from (4.1) that D is independent of $\beta \in \rho(A)$. Then $C\&D$ and S can be split to take their form which is familiar from finite-dimensional systems theory:

$$C\&D \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] = \overline{C} x + D v, \qquad S = \begin{bmatrix} A & B \\ \overline{C} & D \end{bmatrix} \quad (4.9)$$

and we have

$$\mathbf{G}(s) = \overline{C}(sI - A)^{-1} B + D \quad \forall\, s \in \rho(A). \quad (4.10)$$

A system node Σ_{node} is usually associated with the equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \qquad \forall\, t \geqslant 0, \quad (4.11)$$

where S is the system operator of Σ_{node}. Equivalently,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = C\&D \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right], \quad (4.12)$$

for every $t \geqslant 0$.

Definition 4.2 *Let S be a closed linear operator from $X \times U$ to $X \times Y$, with domain $\mathcal{D}(S)$ (but S need not originate from a system node).*

A triple (x, u, y) is called a classical solution *of* (4.11) *on* $[0, \infty)$ *if:*

(a) $x \in C^1([0, \infty); X)$,

(b) $u \in C([0, \infty); U)$, $y \in C([0, \infty); Y)$,

(c) $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \mathcal{D}(S)$ *for all* $t \geqslant 0$,

(d) (4.11) *holds.*

A triple (x, u, y) is called a generalized solution *of* (4.11) *on* $[0, \infty)$ *if*

(e) $x \in C([0, \infty); X)$,

(f) $u \in L^2_{\mathrm{loc}}([0, \infty); U)$, $y \in L^2_{\mathrm{loc}}([0, \infty); Y)$,

(g) *there exists a sequence (x_k, u_k, y_k) of classical solutions of* (4.11) *such that $x_n \to x$ in $C([0, \infty); X)$, $u_k \to u$ in $L^2_{\mathrm{loc}}([0, \infty); U)$, $y_k \to y$ in $L^2_{\mathrm{loc}}([0, \infty); Y)$.*

Here, by $u_k \to u$ in $L^2_{\mathrm{loc}}([0, \infty); U)$ we mean that $\mathbf{P}_{[0,\tau]} u_k \to \mathbf{P}_{[0,\tau]} u$ in $L^2([0, \tau]; U)$ for every $\tau \geqslant 0$, and of course similarly for $y_k \to y$.

We remark that it follows easily from conditions (a)–(d) above that every classical solution of (4.11) on $[0, \infty)$ also satisfies

(h) $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C([0, \infty); \mathcal{D}(S))$,

where the continuity is with respect to the graph norm of S on $\mathcal{D}(S)$. In the case when S is a system node, this graph norm is equivalent to the norm in (4.5).

The following proposition guarantees that for a system node, we have plenty of classical solutions of the system equation (4.11), or equivalently (4.12).

Proposition 4.3 *Let Σ_{node} be a system node on (U, X, Y). If $u \in C^2([0, \infty); U)$ and $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}(C\&D)$, then the equations* (4.12) *have a unique classical solution (x, u, y) satisfying $x(0) = x_0$. Moreover, this classical solution satisfies*

$$x \in C^2([0, \infty); X_{-1}).$$

If u has compact support, then y has a Laplace transform and (2.2) *holds on* $\mathbb{C}_{\omega_0(\mathbb{T})}$.

For the proof we refer to Lemma 4.7.8 in [65] or Proposition 4.2.11 in [73] (various versions of (parts of) this proposition can be found in the literature). Actually, the last sentence of the proposition is not in the cited references, but it is easy to prove.

Let us denote by $\mathcal{D}$ the space of all the pairs $(x_0, u) \in X \times L^2([0,\infty); U)$ which satisfy the assumptions of Proposition 4.3. Notice that $\mathcal{D}$ is dense in $X \times L^2([0,\infty); U)$. Hence, the corresponding space $\mathcal{D}_\tau$ of pairs $(x_0, \mathbf{P}_{[0,\tau]}u)$ is dense in $X \times L^2([0,\tau]; U)$. The last proposition allows us to define the operators Σ_τ from $\mathcal{D}_\tau$ to $X \times L^2([0,\tau], Y)$ such that for any solution of (4.12) and for any $\tau \geqslant 0$,

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix} = \Sigma_\tau \begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix}. \tag{4.13}$$

Definition 4.4 *The system node Σ_{node} is called* well-posed *if for some (hence, for every) $\tau > 0$, the operator Σ_τ from* (4.13) *has a continuous extension*

$$\Sigma_\tau \in \mathcal{L}(X \times L^2([0,\tau], U), X \times L^2([0,\tau], Y)).$$

It is easy to see that Σ_{node} is well-posed iff for some (hence, for every) $\tau > 0$ there is a $c_\tau \geqslant 0$ such that for all classical solutions of (4.12),

$$\begin{aligned} &\|x(\tau)\|_X^2 + \|y\|^2_{L^2([0,\tau]; Y)} \\ &\qquad \leqslant c_\tau^2 \left(\|x(0)\|_X^2 + \|u\|^2_{L^2([0,\tau]; U)} \right). \end{aligned}$$

It is easy to verify that if Σ_{node} is well-posed, then the family $\Sigma = (\Sigma_\tau)_{\tau \geqslant 0}$ is a well-posed linear system as defined in Section 3. Moreover, the operators A, B, C and the transfer function as defined in Section 3 are then the same as defined in this section. Conversely, every well-posed linear system determines a unique well-posed system node, and hence it makes sense to talk about the *combined observation/feedthrough operator* or about the *system operator of a well-posed linear system* (as in (4.3) and (4.7)). If Σ is a well-posed linear system with system operator S, then the dual system (as introduced in Theorem 3.5) has the system operator S^*. (This is not a trivial statement, it is contained in [68, Theorem 3.5].)

Proposition 4.5 *Every well-posed system node is compatible (as defined before* (4.9)*).*

For the proof see [67, Theorem 3.4]. Thus, for well-posed systems we can always find an extension of the observation operator C such that $\overline{C} \in \mathcal{L}(Z, Y)$ and hence, with a suitable operator $D \in \mathcal{L}(U, Y)$, the formulas (4.9) and (4.10) hold. However, $\overline{C}$ (and hence also D) may not be unique. (The operator $C\&D$ is unique.)

For well-posed system nodes Proposition 4.3 can be strengthened. We shall use the following notation: $\mathcal{H}^1_{\rm loc}((0,\infty); U)$ is the space of those functions on $(0,\infty)$ whose restriction to $(0,n)$ is in $\mathcal{H}^1((0,n); U)$, for every $n \in \mathbb{N}$.

Proposition 4.6 *Let Σ_{node} be a well-posed system node on (U, X, Y). Assume that $u \in \mathcal{H}^1_{\rm loc}((0,\infty); U)$ and $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}(C\&D)$. Then the equations* (4.12) *have a unique classical solution (x, u, y) satisfying $x(0) = x_0$. Moreover, we have*

$$\begin{bmatrix} x \\ u \end{bmatrix} \in C([0,\infty); \mathcal{D}(C\&D)), \quad y \in \mathcal{H}^1_{\rm loc}((0,\infty); Y).$$

Using the notation Φ_t, Ψ_∞ and $\mathbb{F}_\infty$ from Section 3*, the functions x, u, y satisfy*

$$x(t) = \mathbb{T}_t x_0 + \Phi_t u, \quad y = \Psi_\infty x_0 + \mathbb{F}_\infty u. \tag{4.14}$$

For the proof see [65, Theorem 4.6.11] or [67, Theorem 3.1]. For inputs of class L^2 we have generalized solutions for (4.12) with additional properties:

Proposition 4.7 *Let Σ_{node} be a well-posed system node on (U, X, Y). If $u \in L^2_{\rm loc}([0,\infty); U)$ and $x_0 \in X$, then the equations in* (4.12) *have a unique generalized solution (x, u, y) satisfying $x(0) = x_0$. Again the functions x, u, y satisfy* (4.14)*. Moreover, x is the unique fuction in $C([0,\infty); X)$ with the property*

$$x(t) = x_0 + \int_0^t [Ax(\sigma) + Bu(\sigma)] \mathrm{d}\sigma \qquad \forall\, t \geqslant 0,$$

the integral being computed in X_{-1}. (This implies that $x \in \mathcal{H}^1_{\rm loc}((0,\infty); X_{-1})$.)

If there exists $\gamma > \omega_0(\mathbb{T})$ such that $u \in L^2_\gamma([0,\infty); U)$, then $y \in L^2_\gamma([0,\infty); Y)$ and the Laplace transforms of u and y satisfy (2.2) *for* $\operatorname{Re} s > \gamma$.

Conversely, with u and x_0 as above, if x and y are given by (4.14) *then (x, u, y) is a generalized solution of* (4.12).

This proposition can be derived with ease from the one before it, combined with the material in [73, Section 4.2] and the material around (3.17).

Let us denote by Σ the well-posed system corresponding to the well-posed system node Σ_{node} (as in Definition 4.4). With the notation of the last proposition, x and y are called the *state trajectory* and the *output function* of Σ_{node} (or of Σ) corresponding to the initial state x_0 and the input function u.

Definition 4.8 *Let U, X and Y be Hilbert spaces. A triple of operators (A, B, C) is called well-posed on (U, X, Y) if there exists a well-posed linear system Σ on (U, X, Y) whose generating triple is (A, B, C).*

This definition is taken from Curtain and Weiss [19]. Clearly, if (A,B,C) is the generating triple of a system node Σ and (A,B,C) is well-posed, then Σ is well-posed. It is useful to have a list of conditions that A, B and C have to satisfy in order to constitute a well-posed triple. The following result was proven in [19].

Proposition 4.9 *A triple of operators (A,B,C) is well-posed on (U,X,Y) if and only if the following conditions are satisfied:*

(1) *A is the generator of an operator semigroup $\mathbb{T}$ on X,*

(2) *$B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for the semigroup $\mathbb{T}$,*

(3) *$C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for the semigroup $\mathbb{T}$,*

(4) *some (hence every) transfer function $\mathbf{G}$ associated with (A,B,C) (i.e., satisfying* (4.1)*) is proper (as defined after* (3.17)*).*

In particular, it follows that if A,B satisfy the conditions (1) and (2) above and C is bounded (i.e., $C \in \mathcal{L}(X,Y)$), then (A,B,C) is well-posed (the properness of $\mathbf{G}(s) = C(sI-A)^{-1}B$ follows from (3.8)). The dual result is that if A,C satisfy (1) and (3) above and B is bounded, then again (A,B,C) is well-posed (this follows from (3.11)). In both of these cases, $\lim_{\alpha\to\infty} \sup_{\operatorname{Re} s>\alpha} \|C(sI-A)^{-1}B\|_{\mathcal{L}(U,Y)} = 0$.

Proposition 4.10 *In Proposition* 4.9 *we may replace the condition* (4) *with:*

(5) *some (hence every) transfer function $\mathbf{G}$ associated with (A,B,C) (i.e., satisfying* (4.1)*) is bounded on a vertical line $\{s \in \mathbb{C} \mid \operatorname{Re} s = \alpha\}$, where $\alpha > \omega_0(\mathbb{T})$.*

Proof. Let A,B,C be operators satisfying the conditions (1), (2) and (3) from Proposition 4.9, as well as condition (5) above. We have to prove that this implies that condition (4) holds. (It is obvious that (4) implies (5).)

Take $s = a+ib \in \mathbb{C}_\alpha$ and denote $z = \alpha + ib$. By integrating on the horizontal segment $[z,s]$ we have, using (4.2),

$$\begin{aligned}\|\mathbf{G}(s)-\mathbf{G}(z)\| &\leqslant \int_\alpha^a \|\mathbf{G}'(x+ib)\|\mathrm{d}x \\ &= \int_\alpha^a \|C((x+ib)I-A)^{-2}B\|\mathrm{d}x.\end{aligned}$$

Choose $\omega \in (\omega_0(\mathbb{T}),\alpha)$. We know from (3.8) and (3.11) that for some $m_1, m_2 > 0$,

$$\|(C(sI-A)^{-1}\|_{\mathcal{L}(X,Y)} \leqslant \frac{m_1}{\sqrt{\operatorname{Re} s - \omega}}\,,$$

$$\|((sI-A)^{-1}B\|_{\mathcal{L}(U,X)} \leqslant \frac{m_2}{\sqrt{\operatorname{Re} s - \omega}}$$

hold for all $s \in \mathbb{C}_\omega$. Combining these with the previous estimate, we get

$$\|\mathbf{G}(s)-\mathbf{G}(z)\| \leqslant \int_\alpha^a \frac{m_1 m_2}{x-\omega}\mathrm{d}x = m_1 m_2 \log\frac{a-\omega}{\alpha-\omega}.$$

From this and the boundedness of $\mathbf{G}$ on the line where $\operatorname{Re} z = \alpha$, it follows that for a suitable $M > 0$ the following (much weaker) estimate holds:

$$\|\mathbf{G}(s)\| \leqslant M e^{\sqrt{|s|}} \qquad \forall\, s \in \mathbb{C}_\alpha.$$

Applying the Phragmen-Lindelöf principle (for a half-plane), see for instance Titchmarsh [71, p. 177], we conclude that $\mathbf{G}$ is bounded on $\mathbb{C}_\alpha$. □

5 Regular linear systems and linear feedback theory

So far, the only formulas to express the output function of a well-posed system (as defined in (4.14)) in terms of A,B,C and $\mathbf{G}$ are (4.12) (in the time domain) or (2.2) (in the frequency domain), and this is not satisfactory, because (4.12) is valid only for classical solutions (see Proposition 4.6), and even then, the operator $C\&D$ is too complicated. We would like to have something simple, like the second equation in (2.1), and we would like it to hold for any input of class L^2, and for almost every time. This cannot be accomplished for every well-posed system, but it works out well for a subclass called regular linear systems. These are systems whose transfer function has a strong limit at $+\infty$ (along the real axis).

Definition 5.1 *Let X and Y be Hilbert spaces, let $\mathbb{T}$ be a strongly continuous semigroup on X and let $C \in \mathcal{L}(X_1, Y)$. The* Λ-extension *of C is the operator*

$$C_\Lambda x_0 = \lim_{\lambda\to+\infty} C\lambda(\lambda I - A)^{-1}x_0,$$

with its domain $\mathcal{D}(C_\Lambda)$ consisting of those $x_0 \in X$ for which the limits exist.

It is easy to see that C_Λ is indeed an extension of C. This extension has various interesting properties, for which we refer to [41,76,78].

Notation. For the remainder of this section, $\Sigma = (\mathbb{T},\Phi,\Psi,\mathbb{F})$ is a well-posed linear system with input space U, state space X and output space Y and $\Sigma_{node} = (A,B,C,\mathbf{G})$ is the corresponding system node, as introduced after Definition 4.4, so that in particular $\mathbf{G}$ is the transfer function of Σ, which is defined for $\operatorname{Re} s > \gamma_{\mathbb{F}}$. We denote by $C\&D$ be the combined observation/feedthrough operator of Σ (or equivalently, of Σ_{node}). χ is the characteristic function of $[0,\infty)$.

Definition 5.2 *For any* $\mathrm{v} \in U$*, the function* $y_{\mathrm{v}} = \mathbb{F}_\infty(\chi \cdot \mathrm{v})$ *is the* step response *of* Σ *corresponding to* v*. The system* Σ *is called* regular *if the following limit exists in* Y*, for every* $\mathrm{v} \in U$*:*

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau y_{\mathrm{v}}(\sigma) \mathrm{d}\sigma = D\mathrm{v}. \tag{5.1}$$

The operator $D \in \mathcal{L}(U, Y)$ *defined by* (5.1) *is called the* feedthrough operator *of* Σ.

This concept was introduced in [77]. Equivalent characterizations of regularity will be given in Theorem 5.6. The following theorem gives the desired simple representation of the output function of a regular linear system.

Theorem 5.3 *If* Σ *is regular, and if we denote the feedthrough operator of* Σ *by* D*, then the output* y *of* Σ *(defined in* (4.14)*) is given by*

$$y(t) = C_\Lambda x(t) + Du(t), \tag{5.2}$$

for almost every $t \geqslant 0$ *(in particular,* $x(t) \in \mathcal{D}(C_\Lambda)$ *for almost every* $t \geqslant 0$*). If* $t \geqslant 0$ *is such that both* u *and* y *are continuous from the right at* t*, then (using those right limits)* (5.2) *holds at* t *(in particular,* $x(t) \in \mathcal{D}(C_\Lambda)$*).*

The proof is in [77], [79] (these papers use another extension of C, denoted C_L, but C_Λ is an extension of C_L, so that Theorem 5.3 follows). Theorem 5.3 implies the following formula for $\mathbb{F}_\infty$ for regular systems:

$$(\mathbb{F}_\infty u)(t) = C_\Lambda \int_0^t \mathbb{T}_{t-\sigma} Bu(\sigma) \mathrm{d}\sigma + Du(t), \tag{5.3}$$

valid for every $u \in L^2_{\mathrm{loc}}([0, \infty); U)$ and almost every $t \geqslant 0$ (in particular, the integral above is in $\mathcal{D}(C_\Lambda)$ for almost every $t \geqslant 0$).

The operators A, B, C and D are called the *generating operators* of Σ, because Σ is completely determined by them via $\dot{x}(t) = Ax(t) + Bu(t)$ and (5.2).

Remark 5.4 Theorem 5.3 has a version for the more general context of well-posed systems. Following [67, Theorem 3.2] we define the Λ-*extension of* $C\&D$ by

$$[C\&D]_\Lambda \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = C_\Lambda \left[x_0 - (\beta I - A)^{-1} B u_0 \right] + \mathbf{G}(\beta) u_0,$$

where $\beta \in \mathbb{C}_{\omega_\mathbb{T}}$ is arbitrary. Its domain $\mathcal{D}([C\&D]_\Lambda)$ consists of those $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in X \times U$ for which $x_0 - (\beta I - A)^{-1} B u_0 \in \mathcal{D}(C_\Lambda)$. Then y (defined in (4.14)) is given by

$$y(t) = [C\&D]_\Lambda \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},$$

for almost every $t \geqslant 0$ (in particular, $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \mathcal{D}([C\&D]_\Lambda)$ for almost every $t \geqslant 0$).

Theorem 5.5 *Assume that* Σ *is regular. Then* $\mathbf{G}$ *is given by*

$$\mathbf{G}(s) = C_\Lambda (sI - A)^{-1} B + D, \qquad \operatorname{Re} s > \omega_0(\mathbb{T})$$

(in particular, $(sI - A)^{-1} BU \subset \mathcal{D}(C_\Lambda)$*).*

The proof of this theorem, as well as of the following one, is given in [79].

We introduce a notation for angular domains in $\mathbb{C}$: for any $\psi \in (0, \pi)$,

$$\mathcal{W}(\psi) = \left\{ re^{i\phi} \mid r \in (0, \infty),\ \phi \in (-\psi, \psi) \right\}.$$

Theorem 5.6 *The following statements are equivalent:*

(1) Σ *is regular, i.e., for every* $\mathrm{v} \in U$ *the limit in* (5.1) *exists.*
(2) *For every* $s \in \rho(A)$ *we have that* $(sI - A)^{-1} BU \subset \mathcal{D}(C_\Lambda)$ *and* $C_\Lambda (sI - A)^{-1} B$ *is an analytic* $\mathcal{L}(U, Y)$*-valued function of* s *on* $\rho(A)$*, uniformly bounded on any half-plane* $\mathbb{C}_\omega$ *with* $\omega > \omega_0(\mathbb{T})$*.*
(3) *There exists* $s \in \rho(A)$ *such that* $(sI - A)^{-1} BU \subset \mathcal{D}(C_\Lambda)$*.*
(4) *Any state trajectory of* Σ *is almost always in* $\mathcal{D}(C_\Lambda)$*.*
(5) *For every* $\mathrm{v} \in U$ *and* $\psi \in \left(0, \frac{\pi}{2}\right)$*,* $\mathbf{G}(s)\mathrm{v}$ *has a limit as* $|s| \to \infty$ *and* $s \in \mathcal{W}(\psi)$*.*
(6) *For every* $\mathrm{v} \in U$*,* $\mathbf{G}(\lambda)\mathrm{v}$ *has a limit as* $\lambda \to +\infty$*, where* $\lambda \in \mathbb{R}$*.*

Moreover, if the limits mentioned in statements (1), (5) *and* (6) *above exist, then they are equal to* $D\mathrm{v}$*, where* D *is the feedthrough operator of* Σ*.*

In view of this theorem, a *regular transfer function* is defined as a proper transfer function that has a strong limit at $+\infty$ (along the real axis). The classical example of a non-regular but proper transfer function is $\mathbf{G}(s) = \cos \log s$, due to Morris [52]. Many more such examples can be found in [68], but we are not aware of a “natural” proper non-regular example stemming from a PDE with some physical meaning.

Example 3.2 (continued). For this example it is easy to establish (using (3.7)) that the generating triple consists of $A = \frac{\mathrm{d}}{\mathrm{d}\zeta}$, $\mathcal{D}(A) = \{x \in \mathcal{H}^1(-h, 0) \mid x(0) = 0\}$, $B = \delta_0$ (the Dirac mass at zero, defined as a functional on $\mathcal{D}(A^*)$ by $\langle \delta_0, \varphi \rangle = \varphi(0)$), $Cx = x(-h)$ for all $x \in \mathcal{D}(A)$ and the transfer function is $\mathbf{G}(s) = e^{-hs}$. It is now clear from the last theorem that this system is regular, having feedthrough operator $D = 0$.

Example 5.7 This is a very old example taken from [19], in which the operators are represented as infinite matrices. Let $X = l^2$ and $U = Y = \mathbb{C}$. Define the operators A, B, C by

$$A = \begin{bmatrix} -1 & & & \\ & -2 & & \\ & & -3 & \\ & & & \ddots \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 1 & \cdots \end{bmatrix},$$

with the natural domain for A. The operators B and C are admissible for the semigroup generated by A, but the triple (A, B, C) is not well-posed. However, if we replace C with

$$C^a = \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots \end{bmatrix},$$

then (A, B, C^a) is well posed and any transfer function associated with this triple is regular. Thus, if we choose $D = 0$, the equations $\dot{x}(t) = Ax(t) + Bu(t)$ and $y(t) = C^a_\Lambda x(t)$ determine a regular linear system. For the proofs see [19, Section 6].

Example 5.8 Consider the following system (taken from [4]) modeling an elastic string occupying a segment $[0, 1]$ under the action of a pointwise force located at $\xi \in (0, 1)$, where the measured output is the velocity of the string at $x = \xi$:

$$\begin{cases} \ddot{w}(x,t) - w_{xx}(x,t) + u(t)\,\delta_\xi = 0, \\ w(0,t) = w(1,t) = 0, \\ w(x,0) = w_0(x),\ \dot{w}(x,0) = w_1(x), \\ y(t) = \dfrac{\mathrm{d}}{\mathrm{d}t} w(\xi, t), \end{cases} \tag{5.4}$$

where δ_ξ is the Dirac mass at ξ and $w(x,t)$ stands for the transverse deflection of the string at the point $x \in (0, 1)$ and time $t \geqslant 0$. The state of the system is $\left[\begin{smallmatrix} w \\ \dot{w} \end{smallmatrix}\right]$.

Proposition 5.9 *Equations* (5.4) *define a regular system with input and output spaces* $U = Y = \mathbb{C}$, *state space* $X = \mathcal{H}^1_0(0,1) \times L^2[0,1]$ *and transfer function*

$$\mathbf{G}(s) = \frac{\sinh(s\xi)\, \sinh[s(\xi - 1)]}{\sinh(s)} \qquad \forall\, s \in \mathbb{C}_0 .$$

Sketch of the proof. We will omit the determination of the operators A, B, C, and concentrate merely on establishing regularity. Assume that $w_0 = 0$ and $w_1 = 0$. Let $\hat{w}$ be the Laplace transform of w with respect to t. It can be easily checked that

$$s^2 \hat{w}(x,s) - \hat{w}_{xx}(x,s) = 0, \tag{5.5}$$

for $x \in (0, \xi) \cup (\xi, 1)$ and $\operatorname{Re} s > 0$,

$$\hat{w}(0,s) = \hat{w}(1,s) = 0 \qquad (\operatorname{Re} s > 0), \tag{5.6}$$

$$[\hat{w}(\cdot,s)]_\xi(s) = 0,\ [\hat{w}_x(\cdot,s)]_\xi = \hat{u}(s), \tag{5.7}$$

where $[g]_\xi$ the jump of the function g at the point ξ. From (5.5) and (5.6) it follows that

$$\hat{w}(x,s) = \begin{cases} K_1 \sinh(\lambda x), & x \in (0, \xi), \\ K_2 \sinh[\lambda(x - 1)], & x \in (\xi, 1), \end{cases}$$

where K_1, K_2 are constants.

Consequently, the solutions of (5.5)-(5.7) have the following form:

$$\hat{w}(x,s) = \begin{cases} \frac{1}{s} \frac{\sinh[s(\xi-1)]\, \sinh(sx)}{\sinh(s)}\, \hat{u}(s), & x \in (0, \xi), \\ \frac{1}{s} \frac{\sinh(s\xi)\, \sinh[s(x-1)]}{\sinh(s)}\, \hat{u}(s), & x \in (\xi, 1). \end{cases}$$

It follows that

$$s\hat{w}(\xi, s) = \mathbf{G}(s)\hat{u}(s),$$

where $\mathbf{G}$ is as given in the proposition. We can easily check that for $\beta > 0$ large enough, $\sup_{\operatorname{Re} s > \beta} |\mathbf{G}(s)| \leqslant 1$, so that $\mathbf{G}$ is proper. It is also easy to check that $\lim_{s \to \infty} \mathbf{G}(s) = -1/2$, so that $\mathbf{G}$ (and hence the system) is regular. □

Remark 5.10 The above example has a natural n-dimensional counterpart, $n \geqslant 2$, where we replace $[0, 1]$ with a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary and we replace the operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with the Dirichlet Laplacian. The point ξ where the control acts is in Ω and the observation is the velocity at ξ. By analogy with the last example, the state space should be $X = \mathcal{H}^1_0(\Omega) \times L^2(\Omega)$ and we should have $X_{-1} = L^2(\Omega) \times \mathcal{H}^{-1}(\Omega)$. However, since $\delta_\xi \in \mathcal{H}^{-s}(\Omega)$ holds only for $s > \frac{n}{2}$, it is easy to verify that the given equations do not correspond to a system node, and we cannot "save" them by changing the state space. In particular, there is no way to formulate these equations as a well-posed system.

Example 5.11 A non-trivial example of regular system described by PDEs in several space dimensions is the wave equation on a bounded domain with smooth boundary, with Dirichlet control and colocated observation:

$$\ddot{z} = \Delta z \ \text{ in } \ \Omega \times (0, \infty), \tag{5.8}$$

$$z = u \quad \text{on } \ \partial\Omega \times (0, \infty), \tag{5.9}$$

$$z(x,0) = z_0(x), \quad \dot{z}(x,0) = w_0(x) \ \text{ for } \ x \in \Omega. \tag{5.10}$$

The input of this system is the function u in (5.9), while the output is

$$y = -\frac{\partial}{\partial \nu}(G\dot{z}) \ \text{ on } \ \partial\Omega \times (0, \infty). \tag{5.11}$$

Here ν is the unit normal vector of $\partial\Omega$ pointing towards the exterior of Ω and the operator $G : \mathcal{H}^{-1}(\Omega) \to \mathcal{H}^1_0(\Omega)$ is defined by

$$Gf = \phi \ \text{ iff } \ \phi \in \mathcal{H}^1_0(\Omega) \text{ and } -\Delta\phi = f,$$

so that, in a certain sense, $G = -\Delta^{-1}$. It has been proved in Ammari [3], using microlocal analysis, that (5.8)–(5.10) determine a well-posed system with input and output space $L^2(\partial\Omega)$ and with state space $X = L^2(\Omega) \times \mathcal{H}^{-1}(\Omega)$. A different proof has been offered in Lasiecka and Triggiani [43]. The fact that this system is regular, with $D = I$, has been established in Guo and Zhang [29]. The method used in [29] combines Fourier analysis and pseudo-differential operators, after a local change of variables reducing the problem to a variable coefficients PDE in a half-space.

In recent years, many other systems described by PDEs in $\mathbb{R}^n$ have been proven to be regular, especially by Bao-Zhu Guo and his collaborators, see [14–17,27–29,86]. They have used advanced PDE techniques to prove regularity for systems involving wave, heat, Euler-Bernoulli beam, elasticity and Schrödinger equations, with constant or variable coefficients. The paper [17] uses Riemannian geometry to prove the regularity of Naghdi's shell equations with boundary control.

At the same time, sophisticated well-posedness and regularity results for systems described by one-dimensional PDEs have been developed in Zwart *et al* [89]. Bounit and Hadd [11] have proved that any well-posed system governed by functional differential equations of neutral type is regular.

Remark 5.12 The weak Λ-extension of C, denoted $C_{\Lambda w}$, is defined similarly as C_Λ, but with a weak limit, so that its domain is larger. *Weak regularity* is defined similarly as regularity, but with a weak limit, see [67]. If Y is finite-dimensional, then of course there is no difference between regularity and weak regularity. Everything we said about regularity and C_Λ (in particular, the three theorems that we have stated up to here in this section) remains valid for the more general concept of weak regularity and for $C_{\Lambda w}$, if we replace limits in the norm of Y with weak limits. It is clear that if a well-posed system is weakly regular, then its dual is also weakly regular. For regular systems, the dual need not be regular, see Example 8.1 in [68]. The main reason why we need the concept of regularity (instead of using just weak regularity) is the feedback theory from [78] and its applications. This theory has substantial parts that we cannot extend to weakly regular systems.

The *degree of unboundedness* of an operator $B \in \mathcal{L}(U, X_{-1})$, denoted $\alpha(B)$, is the infimum of those $\alpha \geqslant 0$ for which there exist positive constants δ, ω such that

$$\|(\lambda I - A)^{-1}B\|_{\mathcal{L}(U,X)} \leqslant \frac{\delta}{\lambda^{1-\alpha}} \quad \forall \lambda \in (\omega, \infty). \quad (5.12)$$

It is clear from (3.8) that for any admissible control operator B we have $\alpha(B) \leqslant \frac{1}{2}$, and if B is bounded then $\alpha(B) = 0$. The *degree of unboundedness* of an operator $C \in \mathcal{L}(X_1, Y)$, denoted $\alpha(C)$, is defined similarly, by replacing in (5.12) $\|(\lambda I - A)^{-1}B\|$ with $\|C(\lambda I - A)^{-1}\|$. Then it follows from (3.11) that for any admissible observation operator C we have $\alpha(C) \leqslant \frac{1}{2}$, and if C is bounded then $\alpha(C) = 0$. The following result is Proposition 4.1 from Curtain and Weiss [20].

Proposition 5.13 *Let Σ be a well-posed linear system with control operator B, observation operator C and transfer function $\mathbf{G}$. If*

$$\alpha(B) + \alpha(C) < 1,$$

then Σ is regular. In fact, $\lim_{\lambda \to +\infty} \mathbf{G}(\lambda)$ exists in the operator norm.

We now recall some static output feedback theory. We use the standing notation of this section, so that Σ is a well-posed system and U, X, Y, A, B, C, $\mathbf{G}(s)$ have their usual meaning. We take a *feedback operator* $K \in \mathcal{L}(Y, U)$ and we are interested in the *closed-loop system* Σ^K that is obtained by imposing the "static output feedback law" $u = Ky + v$, where v is the new input function, as shown in Figure 2. The state and output of Σ^K should be the same as for Σ, as long as their inputs are related by $u = Ky + v$. The trouble with the feedback interconnection from Figure 2 is that it is not necessarily well-posed - sometimes it cannot even be defined. To avoid such situations, we have to introduce the following concept:

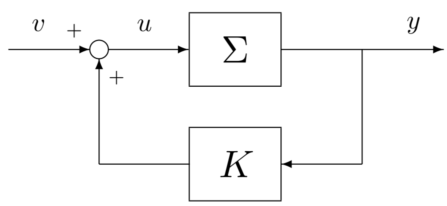

Fig. 2. A well-posed linear system Σ with output feedback via K. If K is admissible, then this is a new well-posed linear system Σ^K, called the closed-loop system.

Definition 5.14 *$K \in \mathcal{L}(Y, U)$ is called an* admissible feedback operator *for Σ (or for $\mathbf{G}$) if $I - \mathbf{G}K$ is invertible on some right half-plane and its inverse is proper.*

In this definition, $I - \mathbf{G}K$ may be replaced equivalently with $I - K\mathbf{G}$. We present some results about well-posed systems with admissible feedback following Weiss [78].

Proposition 5.15 *If K is admissible, then the feedback connection from Figure 2 determines a new well-posed linear system $\Sigma^K = (\Sigma^K_\tau)_{\tau \geqslant 0}$, defined as follows: for each $\tau > 0$, Σ^K_τ is the unique solution of*

$$\Sigma^K_\tau - \Sigma_\tau = \Sigma_\tau \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \Sigma^K_\tau . \quad (5.13)$$

Moreover, the transfer function of Σ^K*, denoted by* $\mathbf{G}^K$*, is given by*

$$\mathbf{G}^K = \mathbf{G}(I - K\mathbf{G})^{-1} = (I - \mathbf{G}K)^{-1}\mathbf{G}$$

and we have the commutation property

$$\Sigma_\tau \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \Sigma_\tau^K = \Sigma_\tau^K \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \Sigma_\tau .$$

We denote by (A^K, B^K, C^K) the generating triple of Σ^K. Note that (unless B is bounded) the domain $\mathcal{D}(A^K)$ may be different from $\mathcal{D}(A)$ and similarly, unless C is bounded, the space X_{-1}^K (the completion of X with respect to the norm $\|x\|_{-1}^K = \|(\beta I - A^K)^{-1}x\|$) may be different from X_{-1}.

Theorem 5.16 *With the above notation, with admissible* K*, the following identities are valid on the right half-plane where* $\mathrm{Re}\ s > \max\{\omega_0(\mathbb{T}), \omega_0(\mathbb{T}^K)\}$*:*

$$[I - \mathbf{G}(s)K]\, C^K (sI - A^K)^{-1} = C(sI - A)^{-1},$$

$$(sI - A^K)^{-1} B^K\, [I - K\mathbf{G}(s)] = (sI - A)^{-1}B .$$

For all $x \in \mathcal{D}(A^K)$ *and for all* $z \in \mathcal{D}(A)$*,*

$$A^K x = \left(A + BKC^K\right) x, \quad Az = \left(A^K - B^K KC\right) z,$$

where in the first formula, A *is regarded as an operator from* X *to* X_{-1}*, and in the second formula,* A^K *is regarded as an operator from* X *to* X_{-1}^K*.*

Theorem 5.17 *With the above notation, with admissible* K*, assume that* Σ *is regular with feedthrough operator* D*. Then* $I - DK$ *(and hence also* $I - KD$*) is left invertible. The closed-loop system* Σ^K *is regular if and only if* $I - DK$ *(and hence also* $I - KD$*) is invertible. In this case, denoting the feedthrough operator of* Σ^K *by* D^K*, the operators* A^K*,* B^K*,* C^K*,* D^K *can be expressed in terms of* A*,* B*,* C*,* D*:*

$$A^K x = \left[A + BK(I - DK)^{-1} C_\Lambda\right] x,$$

$$C^K x = (I - DK)^{-1} C_\Lambda x,$$

for all $x \in \mathcal{D}(A^K)$*, where*

$$\mathcal{D}(A^K) = \{q \in \mathcal{D}(C_\Lambda) \,|\, (A + BK(I - DK)^{-1} C_\Lambda) q \in X\}.$$

Moreover, we have

$$\mathcal{D}(C_\Lambda^K) = \mathcal{D}(C_\Lambda), \qquad C_\Lambda^K = (I - DK)^{-1} C_\Lambda .$$

Regarding the operators B^K *and* D^K *we have*

$$B^K = B(I - KD)^{-1},$$

$$D^K = D(I - KD)^{-1} = (I - DK)^{-1} D .$$

The above formula for B^K is problematic, because B and B^K map into different spaces, so that at first sight it looks like the formula makes no sense. However, there is a natural way to identify a part of X_{-1} with a part of X_{-1}^K, explained in [78, Section 7], and after this identification the formula for B^K makes sense. For other results about the closed-loop system we refer to [78] and also to Staffans [65] and Xu and Weiss [88].

The motivation for introducing regular linear systems has been the simple structure of the output equation (as given in Theorem 5.3) and the simple formula for the transfer function (as given in Theorem 5.5), because these allow us to try to replicate classical ideas from finite-dimensional control theory in an infinite-dimensional context. Good examples of this being done are the papers [81,21] on Luenberger observers, dynamic stabilization and coprime factorization. Regular systems are also used in optimal control, see [18] and the references therein, the theory of exponential stabilization by colocated feedback in [20], in the state feedback regulator theory from [53], in the PI controller theory of [46] (and the references therein) and others. The paper [47] explores the robust stability of feedback systems with respect to small delays in the loop, and here regularity is used not for the reasons mentioned above, but because it enables certain proofs in the frequency domain.

6 Passive linear systems

The concept of passive system and its importance has been recalled in Section 2 (around (2.7)) and this material remains valid in the linear infinite-dimensional context. Many particular quadratic storage functions and supply rates are of interest (see, for instance, [65]) and they may depend on all sorts of weighting operators. In this survey we shall consider only half the norm squared as a storage function, and two particular supply rates, leading to impedance passive systems and to scattering passive ones. Impedance passivity appears more naturally in modeling, but it has a big drawback: it does not imply well-posedness. In finite dimensions this is not an issue, but in infinite-dimensional systems, if we do not have well-posedness, it is sometimes more convenient to transform impedance passive systems into scattering passive ones via the external Cayley transformation described in (2.11).

Definition 6.1 *The system node* $\Sigma_{node} = (A, B, C, \mathbf{G})$ *on* (U, X, Y) *is called* impedance passive *if* $Y = U'$ *(the dual space of* U*) and all the classical solutions of* (4.11) *satisfy, for all* $t \geqslant 0$*,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 \leqslant 2\mathrm{Re}\,\langle u(t), y(t)\rangle_{U,Y} . \tag{6.1}$$

An equivalent condition is that all the generalized solutions of (4.11) satisfy, for every $\tau \geqslant 0$,

$$\|x(\tau)\|^2 - \|x(0)\|^2 \leqslant 2\int_0^\tau \mathrm{Re}\,\langle u(t), y(t)\rangle_{U,Y}\,\mathrm{d}t\,. \quad (6.2)$$

Often U' is identified with U. In finite dimensions, we have already presented this concept at (2.8). Impedance passive systems appear frequently as models of physical systems, and then often $\frac{1}{2}\|x\|^2$ represents the energy of the system and $\mathrm{Re}\,\langle u, y\rangle$ is the power flowing into it. For example, if a component of u is a voltage (or a velocity) then the corresponding component of y is normally a current (or a force).

Theorem 2.4 remains valid in the context of system nodes (in fact it has been proved in this context). The precise statement is the following result, that combines elements from Theorem 4.2, Corollary 4.4 and Theorem 4.6 in Staffans [63].

Theorem 6.2 *If we identify $U' = U$, then Σ_{node} is impedance passive if and only if the operator*

$$T = \begin{bmatrix} A & B \\ -C\&D \end{bmatrix}, \qquad \mathcal{D}(T) = \mathcal{D}(C\&D) \qquad (6.3)$$

is dissipative (equivalently, m-dissipative) on $X \times U$. Moreover, we always have equality in (6.1) *if and only if* $\mathrm{Re}\,\langle T\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right], \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right]\rangle = 0$ *for all* $\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in \mathcal{D}(C\&D)$.

We consider this to be the most significant characterization of impedance passive system nodes. We refer to [1,63,64] for alternative characterizations and related results. The transfer function of an impedance passive system node is always positive, as defined in (2.9), but it need not be proper. It is easy to see (by taking $u = 0$) that the semigroup of an impedance passive system node is always contractive. A generalization of the concept of impedance passive system node has been given in [69]: the idea is to keep the requirement that T is m-dissipative but to drop any other assumption, so that we are no longer dealing with a system node.

Theorem 6.3 *Let $\Sigma_{node} = (A, B, C, \mathbf{G})$ be an impedance passive system node. If $\mathbf{G}$ is bounded on a vertical line in $\mathbb{C}_0$, then Σ_{node} is well-posed.*

This theorem is due to Staffans [63, Theorem 5.1]. The converse is obviously true, if Σ_{node} is well-posed then $\mathbf{G}$ is bounded on any half-plane $\mathbb{C}_\alpha$ with $\alpha > 0$.

We now begin to describe several classes of system nodes with a special structure, that occur often in modeling. These are useful because once we recognize that a system belongs to one of these special classes, we can use readily available results about the class.

Special structure "undamped second order". Let H be a Hilbert space and assume that $A_0 : \mathcal{D}(A_0) \to H$ is positive and boundedly invertible operator. We introduce the scale of Hilbert spaces H_α, $\alpha \in \mathbb{R}$, as follows: for every $\alpha \geqslant 0$, $H_\alpha = \mathcal{D}(A_0^\alpha)$, with the norm $\|z\|_\alpha = \|A_0^\alpha z\|_H$. The space $H_{-\alpha}$ is defined by duality with respect to the pivot space H as follows: $H_{-\alpha} = H_\alpha^*$ for $\alpha > 0$. Equivalently, $H_{-\alpha}$ is the completion of H with respect to the norm $\|z\|_{-\alpha} = \left\|A_0^{-\alpha} z\right\|_H$. The operator A_0 can be extended (or restricted) to each H_α, such that it becomes a bounded operator

$$A_0 : H_\alpha \to H_{\alpha-1} \qquad \forall\, \alpha \in \mathbb{R}.$$

The second ingredient needed for our construction is a bounded linear operator $C_0 : H_{\frac{1}{2}} \to U$, where U is another Hilbert space. We identify U with its dual and we denote $B_0 = C_0^*$, so that $B_0 : U \to H_{-\frac{1}{2}}$. We consider the system described by

$$\ddot{z}(t) + A_0 z(t) = B_0 u(t), \qquad (6.4)$$

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t), \qquad (6.5)$$

where $t \in [0, \infty)$ is the time. The equation (6.4) is understood as an equation in $H_{-\frac{1}{2}}$. Most of the linear equations modelling the undamped vibrations of elastic structures can be written in the form (6.4), where z stands for the displacement field. The state $x(t)$ of this system, its state space X and its semigroup generator $A : H_1 \times H_{\frac{1}{2}} \to X$ are defined by

$$x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}, \quad X = H_{\frac{1}{2}} \times H, \quad A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}. \quad (6.6)$$

The observation operator is $C = \begin{bmatrix} 0 & C_0 \end{bmatrix}$, defined on $\mathcal{D}(A) = X_1 = H_1 \times H_{\frac{1}{2}}$, while $B = C^*$. The operator C has a natural extension $\overline{C} : H_{\frac{1}{2}} \times H_{\frac{1}{2}} \to U$, given by the same formula. It is easy to check that the space Z defined in (4.8) is contained in $H_{\frac{1}{2}} \times H_{\frac{1}{2}}$. Therefore, we can define the $\mathcal{L}(U)$-valued function $\mathbf{G}$ by

$$\mathbf{G}(s) = \overline{C}(sI - A)^{-1}B = sC_0(s^2 I + A_0)^{-1}B_0 \;\; \forall s \in \mathbb{C}_0,$$

and $\mathbf{G}$ is a transfer function associated with the triple (A, B, C), i.e., it satisfies (4.1). It is now easy to see that (6.4) and (6.5) are in fact the system equations (4.12), where $C\&D$ is defined by $C\&D\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] = \overline{C}x$.

Proposition 6.4 *$(A, B, C, \mathbf{G})$ is an impedance passive system node on (U, X, U).*

This can be checked by an easy computation. We mention that in this case T from (6.3) is skew-adjoint on $H_{\frac{1}{2}} \times H \times U$.

The above class of systems has been studied on Ammari and Tucsnak [5] where, in particular, the version of Theorem 6.3 for this class was given. The main focus in [5] was the stabilization for systems in this class using static output feedback. The output feedback stabilization of various systems modeling elastic structures (such as Euler-Bernoulli and Rayleigh beams) has been studied in this framework in Aalto and Malinen [1], [5], Ammari *et al* [4], Guo and Luo [26], Weiss [80] and others.

Special structure "parabolic well-posed". We now move to another simple class of impedance passive systems, which are usually associated to parabolic PDEs. These systems are well-posed and even regular. The construction is as follows: Let H be a Hilbert space and let $A_0 : \mathcal{D}(A_0) \to H$ be a strictly positive operator. For $\alpha \in \mathbb{R}$ we define the scale of spaces H_α as we did after Theorem 6.3. Let U and Y be Hilbert spaces, let $B \in \mathcal{L}(U, H_{-\frac{1}{2}})$ (so that $B^* \in \mathcal{L}(H_{\frac{1}{2}}, U'))$ and let $\overline{C} \in \mathcal{L}(H_{\frac{1}{2}}, Y)$. We denote $A = -A_0$, so that A generates an analytic operator semigroup on H. We denote by C the restriction of $\overline{C}$ to $\mathcal{D}(A) = H_1$. The transfer function associated to A, B and $\overline{C}$ is

$$\mathbf{G}(s) = \overline{C}(sI - A)^{-1}B \qquad \forall\, s \in \mathbb{C}_0$$

and it is easy to see that $\Sigma_{node} = (A, B, C, \mathbf{G})$ is a system node on (U, X, Y).

Proposition 6.5 *With the above notation, Σ_{node} is well-posed and the corresponding well-posed system Σ is regular, with feedthrough operator $D = 0$. Moreover, if z is the state trajectory of Σ corresponding to the initial state $z_0 \in H$ and the input function $u \in L^2_{\rm loc}([0,\infty); U)$, as in Proposition 4.7, then*

$$\|z(t)\|^2 + 2\int_0^t \|z(\sigma)\|^2_{\frac{1}{2}}\,\mathrm{d}\sigma = \|z_0\|^2 + 2\mathrm{Re}\int_0^t \langle u(\sigma), B^* z(\sigma)\rangle_{U,U'}\,\mathrm{d}\sigma \quad (6.7)$$

holds for all $t \geqslant 0$. Thus, if $Y = U'$ and $\overline{C} = B^*$, then Σ is impedance passive.

Proof. Take $z_0 \in H$ and $u \in C^2([0,\infty); U)$ such that $Az_0 + Bu(0) \in H$, which is equivalent to $\left[\begin{smallmatrix} z_0 \\ u(0)\end{smallmatrix}\right] \in \mathcal{D}(C\&D)$. We know from Proposition 4.3 that the equation

$$\dot z(t) = Az(t) + Bu(t), \qquad z(0) = z_0$$

has a unique classical solution which is in $C^1([0,\infty); H)$. This fact and the above differential equation imply that for every $t \geqslant 0$ we have $Az(t) \in H_{-\frac{1}{2}}$ so that $z(t) \in H_{\frac{1}{2}}$. It follows from the above facts that the function $t \mapsto \|z(t)\|^2$ is continuously differentiable and (denoting by $\|B\|$ the norm of B in $\mathcal{L}(U, H_{-\frac{1}{2}})$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2 = 2\mathrm{Re}\,\langle z(t), \dot z(t)\rangle = -2\langle z(t), A_0 z(t)\rangle + 2\mathrm{Re}\,\langle z(t), Bu(t)\rangle_{H_{\frac{1}{2}}, H_{-\frac{1}{2}}} \leqslant -2\|z(t)\|^2_{\frac{1}{2}} + \|z(t)\|^2_{\frac{1}{2}} + \|Bu(t)\|^2_{-\frac{1}{2}} \leqslant -\|z(t)\|^2_{\frac{1}{2}} + \|B\|^2\|u(t)\|^2_U. \quad (6.8)$$

Integrating the above inequality on $[0,\tau]$, we obtain that for every $\tau \geqslant 0$

$$\|z(\tau)\|^2 + \int_0^t \|z(t)\|^2_{\frac{1}{2}}\,\mathrm{d}t \leqslant \|z_0\|^2 + \|B\|^2 \int_0^\tau \|u(t)\|^2_U\,\mathrm{d}t.$$

Denote $y(t) = Cz(t)$. Using that $\|y(t)\| \leqslant \|C\|\|z(t)\|_{\frac{1}{2}}$, the last estimate implies that

$$\|z(\tau)\|^2 + \frac{1}{\|C\|^2}\int_0^\tau \|y(t)\|^2\,\mathrm{d}t \leqslant \|z_0\|^2 + \|B\|^2 \int_0^\tau \|u(t)\|^2_U\,\mathrm{d}t.$$

According to the comments after Definition 4.4, Σ_{node} is well-posed.

Now we prove the regularity of Σ. Denoting

$$\tilde B = A_0^{-1/2}B \in \mathcal{L}(U,H), \quad \tilde C = \overline{C}A_0^{-1/2} \in \mathcal{L}(H,Y),$$

we obtain

$$\mathbf{G}(s) = \tilde C(-A)(sI - A)^{-1}\tilde B.$$

We decompose $-A(sI-A)^{-1} = I - s(sI-A)^{-1}$. The second term is well known to converge strongly to I as $s \to +\infty$ (for real s), so that $\lim_{s\to+\infty} A(sI-A)^{-1}z = 0$ for every $z \in H$. Thus, we have $\lim_{s\to+\infty}\mathbf{G}(s)v = 0$ for every $v \in U$, so that Σ is regular with feedthrough operator zero.

Integrating (6.8) we obtain that (6.7) holds for all pairs $(z_0, u) \in H \times C^2([0,t]; U)$ such that $Az_0 + Bu(0) \in H$. This set of pairs is dense in $H \times L^2([0,t]; U)$, as we have explained before (4.13). Therefore (6.7) holds for all $(z_0, u) \in H \times L^2([0,t]; U)$. In particular, if $Y = U'$ and $\overline{C} = B^*$, then we get (6.2). $\square$

The above result is partially contained in Lemma 3.3 and Theorem 3.1 of Bensoussan *et al* [10]. The regularity part is taken from [14, Section 7].

We mention that the well-posedness part of the last proposition could have been obtained also from Theorem 4.9. Indeed, B and C are admissible according to [73, Proposition 5.1.3] and duality. Finally, the properness of $\mathbf{G}$ follows from the estimate $\|(sI-A)^{-1}\| \leqslant \frac{M}{|s|}$ for analytic semigroups, using the computations from the proof of regularity given above.

We mention that a related class of systems, where A is replaced with iA, has been analyzed in Wen, Chai and Guo [86, Section 5]. For this class of system nodes, they have shown that if the input-output map is bounded, then the system is well-posed and regular, with feedthrough operator zero.

Example 6.6 A system described by PDEs which fits in the above framework is the heat equation on a bounded domain $\Omega \subset \mathbb{R}^n$ with C^2 boundary, with Neumann control and Dirichlet observation. The well-posedness and regularity of this system has been studied in Byrnes *et al* [14]. The equations of the system are

$$\begin{aligned} \dot{z} &= \Delta z \ \text{ in } \ \Omega \times (0, \infty), \\ \frac{\partial z}{\partial \nu} &= u, \quad y = z \quad \text{on } \ \partial\Omega \times (0, \infty), \\ z(x, 0) &= z_0(x) \quad \text{ for } \ x \in \Omega. \end{aligned}$$

The input of this system is u, while the output is y. We choose the state space $X = L^2(\Omega)$, the output space $Y = \mathcal{H}^{\frac{1}{2}}(\partial\Omega)$ and the input space $U = \mathcal{H}^{-\frac{1}{2}}(\partial\Omega)$, so that U is the dual of Y with respect to the pivot space $L^2(\partial\Omega)$, hence $Y = U'$. It has been shown in [14] that, with a suitable definition of A and B, this system fits into the framework of Proposition 6.5, with $\overline{C} = B^*$.

A feedback theory for impedance passive system nodes, which does not fit into the framework of the well-posed feedback theory recalled at the end of Section 3, has been developed in [1]. This theory can handle systems that are composed by interconnecting a finite number of impedance passive systems of boundary control type. Under a certain surjectivity condition, it is shown that the composite system is again an impedance passive system node of boundary control type.

Definition 6.7 *The system node* $\Sigma_{node} = (A, B, C, \mathbf{G})$ *is called* scattering passive *if all the classical solutions of* (4.11) *satisfy, for all* $\geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 \leqslant \|u(t)\|^2 - \|y(t)\|^2. \qquad (6.9)$$

An equivalent condition is that all the generalized solutions of (4.11) satisfy, for every $\tau \geqslant 0$,

$$\|x(\tau)\|^2 + \int_0^\tau \|y(t)\|^2 \,\mathrm{d}t \leqslant \|x(0)\|^2 + \int_0^\tau \|u(t)\|^2 \,\mathrm{d}t. \qquad (6.10)$$

A third equivalent condition is that the operators Σ_τ from (4.13) are *contractions*, so that obviously Σ_{node} is well-posed. Then the corresponding well-posed linear system Σ is called a *scattering passive linear system.* Such systems have been studied in [8,51,63–65,67,69] and other references. (In [51] and [67] such systems were called dissipative.) If Σ is scattering passive then so is its dual Σ^d. Indeed, this follows easily from (3.18), since the operators $\boldsymbol{\mathcal{R}}_\tau$ are unitary. It is clear that the semigroup of a scattering passive system is always contractive. It is also known that its transfer function $\mathbf{G}$ is always Schur, meaning that $\|\mathbf{G}(s)\| \leqslant 1$ for all $s \in \mathbb{C}_0$.

Various characterizations of scattering passive systems have been given in [50,63–65,67–69,82]. These tend to be complicated block operator inequalities, sometimes involving a Cayley transformation. We offer below a criterion that looks simpler, and may be new. It is the infinite-dimensional version of Theorem 2.5 and it is related to the main result of Staffans [66].

Theorem 6.8 *The system node* $\Sigma_{node} = (A, B, C, \mathbf{G})$ *on* (U, X, Y) *is scattering passive (hence well-posed) if and only if the operator*

$$\tilde{T} = \begin{bmatrix} A & B & 0 \\ 0 & -\frac{1}{2} & 0 \\ \multicolumn{2}{l}{C\&D} & -\frac{1}{2} \end{bmatrix}, \quad \mathcal{D}(\tilde{T}) = \mathcal{D}(C\&D) \times Y, \qquad (6.11)$$

is dissipative (equivalently, m-dissipative) on $X \times U \times Y$.

We mention that if it is known that Σ_{node} is compatible, then with the decomposition of $C\&D$ from (4.9) the last line of $\tilde{T}$ becomes $[\overline{C} \ \ D \ -\frac{1}{2}]$. Another remark is that the above theorem does not have a "moreover" part similar to Theorem 6.2, because $\tilde{T}$ cannot possibly have the property $\mathrm{Re}\,\langle \tilde{T}q, q\rangle = 0$ for all $q \in \mathcal{D}(\tilde{T})$.

Proof. The idea of the proof is the same as for Theorem 2.5. We embed Σ_{node} into a larger system node $\widetilde{\Sigma}_{node}$ on $(U \times Y, X, Y \times Y)$ by keeping u (the input signal of Σ_{node}) as the first input, introducing a second input v which has no influence on Σ_{node} and defining two outputs $y_1 = \frac{1}{2}u$ and $y_2 = \frac{1}{2}v - y$, where y is the output of Σ_{node}. Formally, $\widetilde{\Sigma}_{node} = (A, \tilde{B}, \tilde{C}, \tilde{\mathbf{G}})$, where $\tilde{B} = [B \ \ 0]$, $\tilde{C} = \left[\begin{smallmatrix} 0 \\ -C \end{smallmatrix}\right]$ and $\tilde{\mathbf{G}} = \left[\begin{smallmatrix} \frac{1}{2}I & 0 \\ -\mathbf{G} & \frac{1}{2}I \end{smallmatrix}\right]$. The classical solutions of $\widetilde{\Sigma}_{node}$ are precisely the triples $\left(x, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right], \left[\begin{smallmatrix} \frac{1}{2}u \\ \frac{1}{2}v - y \end{smallmatrix}\right]\right)$ where (x, u, y) is a classical solution of Σ_{node} and $v \in C([0, \infty); Y)$.

Suppose that $\tilde{T}$ is dissipative, then it follows from Theorem 6.2 that $\widetilde{\Sigma}_{node}$ is impedance passive. Hence, along classical solutions (x, u, y) of Σ_{node} and for any $v \in C([0, \infty); Y)$ we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 \leqslant 2\mathrm{Re}\left[\langle u(t), \frac{1}{2}u(t)\rangle + \langle v(t), \frac{1}{2}v(t) - y(t)\rangle\right].$$

In particular, by choosing $v = y$ we get precisely (6.9).

Conversely, suppose that (6.9) holds along any classical solution of Σ_{node}. Using that $-\frac{1}{2}\|y_0\|^2 \leqslant \langle v_0, \frac{1}{2}v_0 - y_0\rangle$ for any $v_0, y_0 \in Y$, we obtain that the first estimate in this proof holds, so that $\widetilde{\Sigma}_{node}$ is impedance passive. According to Theorem 6.2 $\tilde{T}$ is m-dissipative. □

Definition 6.9 *The system node* Σ_{node} *is called* scattering energy preserving *if we always have equality in* (6.9) *(or equivalently, in* (6.10)*). The corresponding scattering* *passive system* Σ *is then* scattering energy preserving. *(Thus,* Σ *is scattering energy preserving iff the operators* Σ_τ *are isometric for all* $\tau \geqslant 0$*.)*

Definition 6.10 *The system* Σ *is called* scattering conservative *if both* Σ *and its dual* Σ^d *are scattering energy preserving. (Thus,* Σ *is scattering conservative iff the operators* Σ_τ *are unitary for all* $\tau \geqslant 0$*.) A* scattering conservative system node *is, by definition, a system node that corresponds to a scattering conservative system.*

The above terminology was introduced by Arov and Nudelman [8]. For the theory of conservative systems we refer also to [7,45,50,51,63,68,72,83,84]. In particular, relatively simple necessary and sufficient conditions for a system node to be scattering conservative have been established in [51].

There is an elegant way to transform any impedance passive system node $\Sigma_{\rm imp}$ into a scattering passive one $\Sigma_{\rm sca}$, called the *external Cayley transformation*, informally described by (2.11) and Figure 1. All the discussion in Section 2 remains valid, and Proposition 2.6 remains true, with the obvious modification of writing $[C\&D]_{\rm imp}$ in place of $[C_{\rm imp}\ D_{\rm imp}]$, and similarly for $[C\&D]_{\rm sca}$. We have to be careful to define the domain of $E_{\rm imp}$: this operator now maps $\mathcal{D}([C\&D]_{\rm imp})$ onto $\mathcal{D}([C\&D]_{\rm sca})$. For the proof and a generalization we refer to [69, Section 5].

Special structure "from thin air". Let us explain the origin of the strange name of this class, as used in [72] and [84]. We have announced results about this special structure in our survey [83], when conservative systems were a new and mysterious topic initiated in [8]. It was difficult at that time to find nontrivial examples. We have come across this special structure and we were enthusiastic to have found an easy and unlimited source to produce examples of conservative systems from very simple ingredients, like "out of thin air". It turns out that this class appears naturally in mathematical models of vibrating systems with damping.

Let the Hilbert spaces H and U, the positive operator $A_0 : \mathcal{D}(A_0) \to H$ and the operators $B_0 = C_0^*$ be as for the special structure "undamped second order", discussed a little earlier. We consider the system described by

$$\ddot z(t) + A_0 z(t) + \frac{1}{2} B_0 \frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t) \,=\, B_0 u(t)\,, \qquad (6.12)$$

$$y(t) \,=\, -\frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t) + u(t)\,. \qquad (6.13)$$

Equation (6.12) differs from (6.4) by the presence of the damping term $B_0 \frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t)$, which is sometimes informally written as $B_0 C_0 \dot z(t)$. The state $x(t)$ of this system and its state space X are defined as in (6.6).

For classical solutions, we can rewrite the equations (6.12), (6.13) as a first order system as follows:

$$\begin{cases} \dot x(t) \,=\, Ax(t) + Bu(t)\,, \\ y(t) \,=\, \overline{C} x(t) - u(t)\,, \end{cases} \qquad (6.14)$$

where

$$A \,=\, \begin{bmatrix} 0 & I \\ -A_0 & -\frac{1}{2} B_0 C_0 \end{bmatrix}, \qquad B \,=\, \begin{bmatrix} 0 \\ B_0 \end{bmatrix},$$

$$\mathcal{D}(A) \,=\, \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in H_{\frac12} \times H_{\frac12} \;\middle|\; A_0 z + \frac{1}{2} B_0 C_0 w \in H \right\},$$

$$\overline{C} : H_{\frac12} \times H_{\frac12} \to U\,, \qquad \overline{C} \,=\, [0\ \ C_0]\,.$$

It is not difficult to check that A is m-dissipative. We denote by C the restriction of $\overline{C}$ to $\mathcal{D}(A)$ and for all $s \in \mathbb{C}_0$ we define

$$\begin{aligned} \mathbf{G}(s) &\,=\, \overline{C}(sI - A)^{-1} B - I \\ &\,=\, C_0 s \left(s^2 I + A_0 + \frac{s}{2} B_0 C_0 \right)^{-1} B_0 - I\,. \end{aligned}$$

Proposition 6.11 *With our assumptions,* $(A, B, C, \mathbf{G})$ *is a scattering conservative system node on* (U, X, U)*.*

For the proof we refer to [84]. Many more results about this class of systems and generalizations are in [26,34,50,69,72,80,82,84]. This class can be obtained from the class "undamped second order" via the external Cayley transformation.

Example 6.12 This is a simplified version of the wave equation example appearing in [84, Section 7]. We assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary Γ. Γ_0 and Γ_1 are nonempty open subsets of Γ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\overline{\Gamma_0 \cup \Gamma_1} = \Gamma$. A function $b \in L^\infty(\Gamma_1)$ is given such that $b(x) \neq 0$ for almost every $x \in \Gamma_1$. The equations of the system are

$$\begin{cases} \ddot z(x,t) \,=\, \Delta z(x,t) & \text{on } \Omega \times [0,\infty), \\ z(x,t) \,=\, 0 & \text{on } \Gamma_0 \times [0,\infty), \\ \frac{\partial}{\partial \nu} z(x,t) + |b(x)|^2 \dot z(x,t) & \\ \qquad \,=\, \sqrt{2} \cdot b(x) u(x,t) & \text{on } \Gamma_1 \times [0,\infty), \\ \frac{\partial}{\partial \nu} z(x,t) - |b(x)|^2 \dot z(x,t) & \\ \qquad \,=\, \sqrt{2} \cdot b(x) y(x,t) & \text{on } \Gamma_1 \times [0,\infty), \\ z(x,0) \,=\, z_0(x), \ \ \dot z(x,0) \,=\, w_0(x) & \text{on } \Omega, \end{cases}$$

where u is the input function and y is the output function. The functions z_0 and w_0 are the initial state of the system. The part Γ_0 of the boundary is just reflecting waves, while the active portion Γ_1 is where both the observation and the control take place. We may think of u as the "incoming wave" (which brings energy into the system) and of y as the "outgoing wave".

Define the Hilbert spaces $H = L^2(\Omega)$ and $U = L^2(\Gamma_1)$. For the space $\mathcal{H}^1_{\Gamma_0}(\Omega)$ see [73, Section 13.6]. After a suitable definition of the Neumann trace operator γ_1 on the set Γ_1 (see [84] for the details), we define $A_0 : \mathcal{D}(A_0) \subset L^2(\Omega) \to H$ by

$$A_0 z = -\Delta z,$$
$$\mathcal{D}(A_0) = \{z \in \mathcal{H}^1_{\Gamma_0}(\Omega) \mid \Delta z \in H,\ \gamma_1 z = 0\}.$$

Then A_0 is positive and boundedly invertible. We have $H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}}) = \mathcal{H}^1_{\Gamma_0}(\Omega)$. It was shown in [84] that this system fits into the "from thin air" framework of (6.12) and (6.13). According to Proposition 6.11, this is a conservative linear system with input and output space U and state space $H_{\frac{1}{2}} \times H$. The regularity of this system seems to be an open question.

Special structure "Maxwell". This is a generalization of the structure "from thin air" discussed before. The generalization was needed to fit in Maxwell's equations on a bounded domain, with control and observation from the "active" part of the boundary, and a superconductor in the "reflecting" part of the boundary, with currents in the domain. For lack of space, we refer to [69] and [82] for the details.

7 Well-posed systems with nonlinear feedback

In this section we introduce a well-posedness concept for the closed-loop system obtained from a well-posed linear systems with nonlinear static feedback from its output to one of its inputs. We show that if the nonlinearity satisfies a certain Lipschitz estimate, then the closed-loop system is well-posed.

Let U_1, U_2, X and Y be Hilbert spaces, and $U = U_1 \oplus U_2$. Let $\Sigma^P = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ be a well-posed linear system with input space U, state space X and output space Y. The operators Φ_τ and $\mathbb{F}_\tau$ can be decomposed into blocks according to the above decomposition of U: $\Phi_\tau = [\Phi^1_\tau\ \Phi^2_\tau]$ and $\mathbb{F}_\tau = [\mathbb{F}^1_\tau\ \mathbb{F}^2_\tau]$. The transfer function of Σ^P can be decomposed similarly: $\mathbf{G} = [\mathbf{G}_1\ \mathbf{G}_2]$. As before, $\omega_0(\mathbb{T})$ is the growth bound of $\mathbb{T}$.

Let $\mathcal{N} : Y \to U_2$ be a Lipschitz map with Lipschitz constant L. The feedback interconnection of Σ^P and $\mathcal{N}$, denoted by $\Sigma^{\mathcal{N}}$, is the dynamic system obtained by imposing that u_2, the second component of u, is obtained from y via $\mathcal{N}$:

$$u_2(t) = \mathcal{N} y(t) \qquad \forall\, t \in [0, \infty).$$

The system $\Sigma^{\mathcal{N}}$, shown as a block diagram in Figure 3, is said to be *well-posed* if for any input $u_1 \in L^2_{\rm loc}([0,\infty); U_1)$ and any initial state $z_0 \in X$, there exist unique functions $z \in C([0,\infty); X)$ (the state trajectory) and $y \in L^2_{\rm loc}([0,\infty); Y)$ (the output function) that satisfy

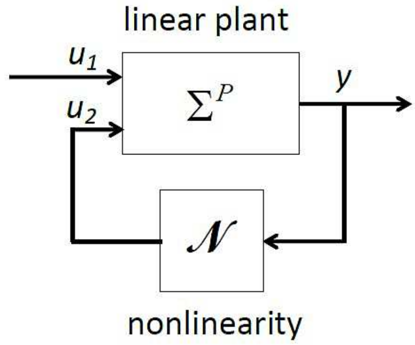

Fig. 3. The nonlinear infinite-dimensional system $\Sigma^{\mathcal{N}}$ obtained from the well-posed linear system Σ^P by static output feedback through the static nonlinearity $\mathcal{N}$.

$$z(t) = \mathbb{T}_t z_0 + \Phi^1_t u_1 + \Phi^2_t \mathcal{N} y, \qquad (7.1)$$
$$\mathbf{P}_t y = \Psi_t z_0 + \mathbb{F}^1_t u_1 + \mathbb{F}^2_t \mathcal{N} y, \qquad (7.2)$$

for all $t \geqslant 0$, and moreover, on any bounded time interval $[0, \tau]$, $z(\tau)$ and $\mathbf{P}_\tau y$ depend continuously on z_0 and on $\mathbf{P}_\tau u_1$. The continuous dependence is meant with respect to the usual norm for states, and with respect to the L^2 norm for the input and output functions.

Well-posed linear systems with nonlinear feedback, more or less in the above framework, have been studied in Logemann and Ryan [48] and in more detail in Jayawardhana, Logemann and Ryan [39]. A relevant earlier reference is Jacob, Dragan and Pritchard [33]. However, it seems that the following straightforward theorem is not available in the cited sources.

Remark 7.1 Suppose that the system $\Sigma^{\mathcal{N}}$ from (7.1)-(7.2) is well-posed and let z be the state trajectory corresponding to the initial state z_0 and the input function u_1. We denote by A the semigroup generator of Σ^P and by $B = [B_1\ B_2]$ its control operator. Then for every $t \geqslant 0$ we have

$$z(t) - z_0 = \int_0^t [A z(\sigma) + B_1 u_1(\sigma) + B_2 \mathcal{N}(y(\sigma))]\, {\rm d}\sigma,$$

and the function under the integral takes values in $L^2_{\rm loc}([0,\infty); X_{-1})$. Indeed, this follows by applying Proposition 4.7 to $(x, \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right], y)$.

Theorem 7.2 *With the notation of this section, if*

$$\inf_{\omega > \omega_0(\mathbb{T})} \|\mathbb{F}^2_\infty\|_\omega L < 1,$$

where L is the Lipschitz constant for the nonlinearity, then $\Sigma^{\mathcal{N}}$ is well-posed.

Proof. Fix $T > 0$ and consider the output equation (7.2) on the interval $[0, T]$, where we denote $y_T = \mathbf{P}_T y$ and use the causality of $\mathbb{F}_T$:

$$y_T = \Psi_T z_0 + \mathbb{F}_T^1 u_1 + \mathbb{F}_T^2 \mathcal{N} y_T . \tag{7.3}$$

Fix $\omega > \omega_0(\mathbb{T})$ such that $\|\mathbb{F}_\infty^2\|_\omega L = a < 1$. Define the nonlinear map M_T from $L^2_\omega([0,T];Y)$ to itself by

$$M_T(z) = \Psi_T z_0 + \mathbb{F}_T^1 u_1 + \mathbb{F}_T^2 \mathcal{N} z .$$

For each $z \in L^2_\omega([0,T];Y)$, since $\mathbb{F}_T^2 z = \mathbf{P}_T \mathbb{F}_\infty^2 z$, we have $\|\mathbb{F}_T^2 z\|_{L^2_\omega} \leqslant \|\mathbb{F}_\infty^2\|_\omega \|z\|_{L^2_\omega}$. It follows that for any $y_1, y_2 \in L^2_\omega([0,T];Y)$,

$$\begin{aligned}\|M_T(y_1) - M_T(y_2)\|_{L^2_\omega} &= \|\mathbb{F}_T^2 \mathcal{N} y_1 - \mathbb{F}_T^2 \mathcal{N} y_2\|_{L^2_\omega} \\ &\leqslant a\|y_1 - y_2\|_{L^2_\omega} .\end{aligned}$$

Thus, M_T is a strict contraction on $L^2_\omega([0,T];Y)$. According to the contraction mapping theorem (see the survey [13]) M_T has a unique fixed point $y_T \in L^2_\omega([0,T];Y)$ which satisfies (7.3). The above reasoning works for any $T > 0$, so that in fact we get a family of functions y_T, each satisfying (7.3).

The continuous dependence of y_T on both z_0 and $\mathbf{P}_T u_1$ follows from Theorem 3.8 in [13] which states the following: The fixed points of a family of contractions that depend continuously on a parameter λ, belonging to a metric space, and having a uniform contraction constant, depend continuously on λ.

We claim that for any $\tau > T > 0$,

$$y_T = \mathbf{P}_T y_\tau . \tag{7.4}$$

To prove this, we apply $\mathbf{P}_T$ to (7.3) in which we have used τ in place of T, obtaining

$$\mathbf{P}_T y_\tau = \Psi_T z_0 + \mathbb{F}_T^1 u_1 + \mathbb{F}_T^2 \mathcal{N} y_\tau .$$

Using causality, this becomes

$$\mathbf{P}_T y_\tau = \Psi_T z_0 + \mathbb{F}_T^1 u_1 + \mathbb{F}_T^2 \mathcal{N} \mathbf{P}_T y_\tau .$$

Since the solution of (7.3) is unique, we get (7.4).

It follows from (7.4) that there exists a unique $y \in L^2_{\mathrm{loc}}([0,\infty);Y)$ such that $\mathbf{P}_t y = y_t$ for each $t > 0$. Once y is defined, the function z can be computed directly from (7.1). The continuous dependence of $z(T)$ on z_0 and $\mathbf{P}_T u_1$ is a consequence of the continuous dependence of y_T on the same parameters (as shown earlier). $\square$

Remark 7.3 It follows from the last theorem that if the transfer function $\mathbf{G}_2(s)$ decays to zero as $\mathrm{Re}\, s \to \infty$, uniformly in $\mathrm{Im}\, s$, then $\Sigma^{\mathcal{N}}$ is well-posed irrespective of L. This is the case, for instance, if either the control operator B_2 or the observation operator C of Σ^P is bounded and its feedthrough operator from u_2 to y is zero. This follows from our comments preceding Proposition 4.10.

A class of well-posed systems with bilinear feedback. Linear systems with nonlinear feedback that depends both on the output and the state appear in the modeling of some physical systems, for instance systems involving fluids. We shall now examine a class of nonlinear systems $\Sigma^{\mathcal{N}}$ described by

$$\begin{aligned} z(t) &= \mathbb{T}_t z_0 + \Phi_t^1 u_1 + \Phi_t^2 \mathcal{N}(z,y), && (7.5)\\ \mathbf{P}_t y &= \Psi_t z_0 + \mathbb{F}_t^1 u_1 + \mathbb{F}_t^2 \mathcal{N}(z,y), && (7.6)\end{aligned}$$

where the well-posed linear system Σ^P is as at the beginning of this section and $\mathcal{N} : X \times Y \to U_2$ is continuous. These equations resemble (7.1), (7.2). Often $\mathcal{N}$ is a continuous bilinear function (such as in the Navier-Stokes or Burgers equations). The equations correspond to taking the feedback $u_2(t) = \mathcal{N}(z(t), y(t))$ for Σ^P.

The *local well-posedness* of the closed-loop system $\Sigma^{\mathcal{N}}$ means that for every $M > 0$ there exists $T > 0$ such that: for any $u_1 \in L^2([0,\infty);U_1)$ and $z_0 \in X$ with $\|z_0\| + \|u_1\|_{L^2([0,\infty);U_1)} \leqslant M$, there exist unique functions $z \in C([0,T];X)$ (the state trajectory) and $y \in L^2([0,T];Y)$ (the output function) that satisfy (7.5) and (7.6) for all $t \in [0,T]$. Moreover, $z(t)$ (for $t \in [0,T]$) and $\mathbf{P}_{[0,T]} y$ depend continuously on z_0 and on $\mathbf{P}_{[0,T]} u_1$. (For $\mathbf{P}_{[0,T]} u_1$ and for $\mathbf{P}_{[0,T]} y$ we use the L^2 norm.) The *well-posedness* of $\Sigma^{\mathcal{N}}$ means that in the above definition, for every $M > 0$ we may take any $T > 0$. This can then be reformulated in a similar way as (7.1) and (7.2).

Remark 7.4 Remark 7.1 can easily be reformulated for the system from (7.5), (7.6), assuming local well-posedness and replacing $\mathcal{N}(y)$ with $\mathcal{N}(z,y)$.

Remark 7.5 For a locally well-posed system we have a property similar to Corollary 2.3. Suppose that for some $z_0 \in X$, $u_1 \in L^2_{\mathrm{loc}}([0,\infty);U_1)$ and $\delta > 0$, $[0,\delta)$ is the maximal interval of existence of the solution of (7.5), (7.6) with $z(0) = z_0$. Then for every $c > 0$ there exists $T \in [0,\delta)$ such that $\|z(T)\| \geqslant c$. Indeed if there were a $c > 0$ such that $\|z(t)\| \leqslant c$ for all $t \in [0,\delta)$ then, according to the definition of local well-posedness, we could extend the solution beyond δ.

Theorem 7.6 *We assume that $Y \subset X$, with continuous and dense imbedding, that $\mathcal{N} : X \times Y \to U_2$ is bilinear, continuous and that there exists $K \geqslant 0$ and $p \in (0,1)$ such that*

$$\|\mathcal{N}(z,y)\|_{U_2} \leqslant K\|z\|_X \, \|y\|_X^{1-p} \, \|y\|_Y^p, \tag{7.7}$$

for every $z \in X$ and $y \in Y$. Moreover, assume that C admits an extension $C \in \mathcal{L}(X)$ and the system is such that its output is given by $y(t) = Cz(t)$. Then the closed-loop system $\Sigma^{\mathcal{N}}$ from (7.5) and (7.6) is locally well-posed.

Proof. The first step of the proof is to introduce the nonlinear loop gain operator $\mathcal{G}_T$ and to prove an estimate for it (estimate (7.11) below). Let $u_1 \in L^2([0,\infty);U_1)$ be fixed, let $T > 0$ and let $\mathcal{G}_T : L^2([0,T];U_2) \to L^2([0,T];U_2)$ be defined by

$$[\mathcal{G}_T(v)](t) = \mathcal{N}(z(t), y(t)) \tag{7.8}$$

for all $t \in [0,T]$, $v \in L^2([0,T];U_2)$, where

$$z(t) = \mathbb{T}_t z_0 + \Phi_t^1 u_1 + \Phi_t^2 v,$$

$$\mathbf{P}_{[0,T]} y = \Psi_T z_0 + \mathbb{F}_T^1 u_1 + \mathbb{F}_T^2 v.$$

It follows from the continuity of z and of $\mathcal{N}$ that indeed $\mathcal{G}_T(v) \in L^2([0,T];U_2)$.

The existence and uniqueness result in the statement is clearly equivalent to the existence and uniqueness of a fixed point of $\mathcal{G}_T$. In order to solve this fixed point problem, we note that (7.7) implies that for almost every $t \in [0,T]$,

$$\|\mathcal{G}_T(v)(t)\|_{U_2}^{\frac{2}{p}} \leqslant K^{\frac{2}{p}} \|C\|_{\mathcal{L}(X)}^{\frac{2(1-p)}{p}} \|z(t)\|_X^{\frac{4-2p}{p}} \|y(t)\|_Y^2 .$$

Using the uniform boundedness of the operators $\mathbb{T}_t$, Φ_t^1 and Φ_t^2 on the interval $[0,T]$, we have that there exists $k_T > 0$, non-decreasing with respect to T, such that

$$\|z(t)\| \leqslant k_T \left(\|z_0\| + \|u_1\|_{L^2([0,T];U_1)} + \|v\|_{L^2([0,T];U_2)}\right). \tag{7.9}$$

From the last two estimates we obtain that

$$\|\mathcal{G}_T(v)\|_{L^{\frac{2}{p}}([0,T];U_2)}^{\frac{2}{p}} \leqslant k_{1,T} \left(\|z_0\| + \|u_1\|_{L^2([0,T];U_1)} + \|v\|_{L^2([0,T];U_2)}\right)^{\frac{4-2p}{p}} \|y\|_{L^2([0,T];Y)}^2,$$

where $k_{1,T}$ is non decreasing with respect to T. Using the boundedness of the operators Ψ_T, $\mathbb{F}_T^1$ and $\mathbb{F}_T^2$, we have that there exists $h_T \geqslant 0$, non-decreasing with respect to T, such that

$$\|y\|_{L^2([0,T];Y)} \leqslant h_T \left(\|z_0\| + \|u_1\|_{L^2([0,T];U_1)} + \|v\|_{L^2([0,T];U_2)}\right). \tag{7.10}$$

Combining the last two estimates we obtain that

$$\|\mathcal{G}_T(v)\|_{L^{\frac{2}{p}}([0,T];U_2)}^{\frac{2}{p}} \leqslant k_{2,T} \left(\|z_0\| + \|u_1\|_{L^2([0,T];U_1)} + \|v\|_{L^2([0,T];U_2)}\right)^{\frac{4}{p}}, \tag{7.11}$$

where $k_{2,T}$ is non-decreasing with respect to T.

The second step in the proof is to show that $\mathcal{G}_T$ leaves certain balls in $L^2([0,T];U_2)$ invariant. Let $M > 0$ and assume that

$$\|z_0\| + \|u_1\|_{L^2([0,T];U_1)} \leqslant M. \tag{7.12}$$

We denote by $\mathcal{B}_{M,T}$ the ball of radius M, centered at the origin, in $L^2([0,T];U_2)$. We claim that, for T sufficiently small, this ball is invariant under $\mathcal{G}_T$. Indeed, using (7.11) it follows that for $v \in \mathcal{B}_{M,T}$ we have

$$\|\mathcal{G}_T(v)\|_{L^{\frac{2}{p}}([0,T];U_2)} \leqslant k_{2,T}^{\frac{p}{2}} (2M)^2. \tag{7.13}$$

It is easy to see from Hölder's inequality that for any $f \in L^{\frac{2}{p}}[0,T]$ we have

$$\|f\|_{L^2[0,T]} \leqslant \|f\|_{L^{\frac{2}{p}}[0,T]} T^{\frac{1-p}{2}}. \tag{7.14}$$

Applying this for $f(t) = \|[\mathcal{G}_T(v)](t)\|$ we obtain from (7.13) that for every $v \in \mathcal{B}_{M,T}$ we have

$$\|\mathcal{G}_T(v)\|_{L^2([0,T];U_2)} \leqslant k_{2,T}^{\frac{p}{2}} (2M)^2 \, T^{\frac{1-p}{2}}.$$

Since $k_{2,T}$ is non-decreasing in T, it is clear from the above that for T sufficiently small (depending on the system and on M) we have $\mathcal{G}_T(v) \in \mathcal{B}_{M,T}$ for every $v \in \mathcal{B}_{M,T}$.

The third step is to show that for every $M > 0$ there exists a $T > 0$ such that if z_0 and u_1 satisfy (7.12) then, besides leaving $\mathcal{B}_{M,T}$ invariant, $\mathcal{G}_T$ is a strict contraction on $\mathcal{B}_{M,T}$. Let v_1, $v_2 \in \mathcal{B}_{M,T}$. Then

$$\begin{aligned}\mathcal{G}_T(v_1) - \mathcal{G}_T(v_2) &= \mathcal{N}(z_1, y_1) - \mathcal{N}(z_2, y_2) \\ &= \mathcal{N}(z_1 - z_2, y_1) + \mathcal{N}(z_2, y_1 - y_2),\end{aligned} \tag{7.15}$$

where for $j \in \{1,2\}$,

$$z_j(t) = \mathbb{T}_t z_0 + \Phi_t^1 u_1 + \Phi_t^2 v_j,$$

$$\mathbf{P}_{[0,T]} y_j = \Psi_T z_0 + \mathbb{F}_T^1 u_1 + \mathbb{F}_T^2 v_j.$$

From (7.15) and (7.7) it follows that

$$\begin{aligned}&\|\mathcal{G}_T(v_1) - \mathcal{G}_T(v_2)\|_{L^{\frac{2}{p}}([0,T];U_2)} \\ &\leqslant K \left[\int_0^T \|z_1 - z_2\|_X^{\frac{2}{p}} \|y_1\|_X^{\frac{2-2p}{p}} \|y_1\|_Y^2 \,\mathrm{d}t\right]^{\frac{p}{2}} \\ &+ K \left[\int_0^T \|z_2\|_X^{\frac{2}{p}} \|y_1 - y_2\|_X^{\frac{2-2p}{p}} \|y_1 - y_2\|_Y^2 \,\mathrm{d}t\right]^{\frac{p}{2}}.\end{aligned} \tag{7.16}$$

To estimate the first term in right-hand side of the above inequality we note that by applying (7.9) it follows that

$$\|z_1(t) - z_2(t)\|_X \leqslant k_T \|v_1 - v_2\|_{L^2([0,T];U_2)}, \tag{7.17}$$

for every $t \in [0,T]$. Combining the last inequality and (7.9), (7.10) it follows that

$$\begin{aligned}&\left[\int_0^T \|z_1(t) - z_2(t)\|_X^{\frac{2}{p}} \|y_1(t)\|_X^{\frac{2-2p}{p}} \|y_1(t)\|_Y^2 \,\mathrm{d}t\right]^{\frac{p}{2}} \\ &\leqslant k_{3,T} \|z_1 - z_2\|_{C([0,T];X)} M^{1-p} \|y_1\|_{L^2([0,T];Y)}^p \\ &\leqslant k_{4,T} M \|v_1 - v_2\|_{L^2([0,T];U_2)},\end{aligned} \tag{7.18}$$

with $k_{3,T}$ and $k_{4,T}$ non decreasing with respect to T. Using again (7.9), (7.10) and (7.17), the last term on the right-hand side of (7.16) satisfies

$$\left[\int_0^T \|z_2(t)\|_X^{\frac{2}{p}} \|y_1(t) - y_2(t)\|_X^{\frac{2-2p}{p}} \|y_1(t) - y_2(t)\|_Y^2 \,\mathrm{d}t\right]^{\frac{p}{2}}$$
$$\leqslant k_{5,T} M \|v_1 - v_2\|_{L^2([0,T];U_2)}^{1-p} \|y_1 - y_2\|_{L^2([0,T];Y)}^p$$
$$\leqslant k_{6,T} M \|v_1 - v_2\|_{L^2([0,T];U_2)}, \quad (7.19)$$

with $k_{5,T}$ and $k_{6,T}$ non-decreasing with respect to T. By using (7.16)-(7.19), together with (7.9) and (7.10), it follows that for every z_0 and u_1 satisfying (7.12) and v_1, $v_2 \in \mathcal{B}_{M,T}$ we have

$$\|\mathcal{G}_T(v_1) - \mathcal{G}_T(v_2)\|_{L^{\frac{2}{p}}([0,T];U_2)} \leqslant k_{7,T} M \|v_1 - v_2\|_{L^2([0,T];U_2)},$$

with $k_{7,T}$ non-decreasing with respect to T. Using above (7.14) it follows that

$$\|\mathcal{G}_T(v_1) - \mathcal{G}_T(v_2)\|_{L^2([0,T];U_2)} \leqslant k_{7,T} M T^{\frac{1-p}{2}} \|v_1 - v_2\|_{L^2([0,T];U_2)},$$

for every z_0 and u_1 satisfying (7.12) and v_1, $v_2 \in \mathcal{B}_{M,T}$. We have thus shown that for every $M > 0$ there exists $T > 0$ such that for z_0 and u_1 satisfying (7.12) the map $\mathcal{G}_{M,T}$ is a strict contraction of $\mathcal{B}_{M,T}$. Consequently, for every $M > 0$ there exists $T > 0$ such that for z_0 and u_1 satisfying (7.12) the map $\mathcal{G}_{M,T}$ admits a unique fixed point, depending continuously on z_0 and u_1. In other words, we have obtained the conclusion that $\Sigma^{\mathcal{N}}$ is locally well-posed. □

8 An example with the Burgers equation

In this section we use the abstract nonlinear feedback theory from Section 7 to prove the well-posedness of the Burgers equation with distributed control. The results here are not new, only the approach. For the study and control of Burgers equation see Ly *et al* [49], Krstic [42] and the references therein. All the function spaces (such as Sobolev spaces) in this section contain only real-valued functions.

Consider the following system with state trajectory z, control input u and output function y:

$$\begin{cases} \dot{z} = z_{xx} - z z_x + u_1 & t \geqslant 0,\ x \in (0,1), \\ z(t,0) = z(t,1) = 0 & t \geqslant 0, \\ z(0,x) = z_0(x) & y \in (0,1), \\ y(t,x) = z(t,x) & t \geqslant 0,\ x \in (0,1). \end{cases} \quad (8.1)$$

The main result in this section is the following. We use the notation $\mathcal{H}^1_{\mathrm{loc}}$ as introduced before Proposition 4.6.

Theorem 8.1 *For every $z_0 \in \mathcal{H}^1_0(0,1)$ and $u_1 \in L^2_{\mathrm{loc}}([0,\infty); L^2[0,1])$, there exists a unique solution z of* (8.1) *such that*

$$z \in \mathcal{H}^1_{\mathrm{loc}}((0,\infty); L^2[0,1]) \cap C([0,\infty); \mathcal{H}^1_0(0,1)) \cap L^2_{\mathrm{loc}}([0,\infty); \mathcal{H}^2(0,1)), \quad (8.2)$$

and the first line in (8.1) *holds in $L^2_{\mathrm{loc}}([0,\infty); L^2[0,1])$.*

To prove this theorem, first we introduce a well-posed linear system Σ^P as follows. We introduce the state, input and output spaces

$$X = \mathcal{H}^1_0(0,1), \quad U_1 = U_2 = L^2(0,1),$$
$$Y = \mathcal{H}^2(0,1) \cap \mathcal{H}^1_0(0,1),$$

and the operator $A : \mathcal{D}(A) \to X$ by

$$A\varphi = \varphi_{xx} \qquad \forall\, \varphi \in \mathcal{D}(A),$$
$$\mathcal{D}(A) = \left\{\varphi \in \mathcal{H}^3(0,1) \ \mid\ \varphi,\ \varphi_{xx} \in \mathcal{H}^1_0(0,1)\right\}.$$

It is well known (see, for instance, Proposition 3.5.1 and the beginning of Section 6.7 in [73]) that $A < 0$, so that A generates an analytic operator semigroup $\mathbb{T}$ on X. Moreover, the corresponding space $X_{\frac{1}{2}} = \mathcal{D}((-A)^{\frac{1}{2}})$ (endowed with the graph norm of $(-A)^{\frac{1}{2}}$) and its dual with respect to the pivot space X are

$$X_{\frac{1}{2}} = \mathcal{H}^2(0,1) \cap \mathcal{H}^1_0(0,1), \quad X_{-\frac{1}{2}} = L^2[0,1],$$

see Sections 3.4 and 3.5 in [73]. Consider the control operators $B_k \in \mathcal{L}(U_k, X_{-\frac{1}{2}})$ defined by $B_1 = B_2 = I$ (the identity operator on $L^2[0,1]$). For $t > 0$ we denote Φ^k_t, with $k \in \{1,2\}$, the corresponding input maps, as in (3.6) (we have $\Phi^1 = \Phi^2$).

We denote by C the identity operator of $\mathcal{H}^1_0(0,1)$, which can be restricted to an unbounded observation operator $C \in \mathcal{L}(X_1, Y)$. We set $(\Psi_\infty z)(t) = C\mathbb{T}_t z$ for every $z \in X_1$ and $t \geqslant 0$, as in (3.10) and $\Psi_\tau = \mathbf{P}_{[0,\tau]} \Psi_\infty$. For $t > 0$ we denote by $\mathbb{F}^k$, with $k \in \{1,2\}$, the input-output maps defined by $(\mathbb{F}^k u)(t) = C\Phi^k_t u$ (we have $\mathbb{F}^1 = \mathbb{F}^2$). We denote $\Phi = [\Phi^1\ \Phi^2]$ and $\mathbb{F} = [\mathbb{F}^1\ \mathbb{F}^2]$.

Proposition 8.2 *We have that $\Sigma^P = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is a regular linear system. If z is the state trajectory of Σ^P corresponding to the initial state z_0 and the input functions u_1 and u_2 and y is the corresponding output function, then*

$$\frac{1}{2}\|z(t)\|^2_{L^2} = \frac{1}{2}\|z_0\|^2_{L^2} - \int_0^t \|z_x(\sigma)\|^2_{L^2} \,\mathrm{d}\sigma + \int_0^t \langle (u_1(\sigma) + u_2(\sigma), y(\sigma)\rangle_{L^2} \,\mathrm{d}\sigma. \quad (8.3)$$

Proof. The fact that Σ^P is regular follows from Proposition 6.5 with $H = X$.

To prove (8.3) we apply the same Proposition 6.5 but with $H = X_{-\frac{1}{2}} = L^2[0,1]$ and U, B as before. Then the identity (6.7) becomes (8.3). □

With the above notation, we show below that the system (8.1) has the form (7.5), (7.6), where the bilinear and continuous operator $\mathcal{N} : X \times Y \to U_2$ is defined by

$$\mathcal{N}(z, y) = -zy_x . \qquad (8.4)$$

The abstract version of Theorem 8.1 is the following:

Theorem 8.3 *With the specified spaces and operators, the system defined by* (7.5), (7.6) *is well-posed.*

A first step in proving Theorem 8.3 is the following:

Proposition 8.4 *The system considered in Theorem* 8.3 *is locally well-posed. For some $z_0 \in X$ and $u_1 \in L^2([0,\infty); U_1)$, let $[0, T)$ be the maximal interval of existence of the solution of* (7.5), (7.6)*. Then for every $t \in [0, T)$,*

$$\frac{1}{2}\|z(t)\|^2_{L^2} = \frac{1}{2}\|z_0\|^2_{L^2} - \int_0^t \|z_x(\sigma)\|^2_{L^2}\,\mathrm{d}\sigma + \int_0^t \langle u_1(\sigma), y(\sigma)\rangle_{L^2}\,\mathrm{d}\sigma . \qquad (8.5)$$

Proof. In order to apply Theorem 7.6 we first note that from (8.4) it follows that

$$\|\mathcal{N}(z, y)\|_{U_2} = \|zy_x\|_{L^2[0,1]} \leqslant \|z\|_{C[0,1]}\,\|y_x\|_{L^2[0,1]} \leqslant K_0\|z\|_X\|y\|_X ,$$

so that (7.7) holds with $p = 0$. Since the embedding $Y \subset X$ is continuous, it follows that (7.7) holds for any $p \in (0, 1)$. From Theorem 7.6 we obtain the local well-posedness of the system.

To prove (8.5), we take $u_2 = \mathcal{N}(z, y)$ in (8.3). Using that $\int_0^1 z_x(t, x)z^2(t, x)\,\mathrm{d}x = \frac{1}{3}\left[z^3(t, 1) - z^3(t, 0)\right] = 0$, we obtain (8.5). □

Proof of Theorem 8.3. We have seen in Proposition 8.4 that the system $\Sigma^{\mathcal{N}}$ is locally well-posed. Let $z_0 \in X$, $u_1 \in L^2_{\rm loc}([0,\infty); U_1)$ and let z be the corresponding solution of (7.5), (7.6), which is defined on the maximal interval $[0, T)$. We assume that T is finite and this will lead to a contradiction.

From the well-posedness of the system Σ^P (see Proposition 8.2) and the fact that $u_2 = \mathcal{N}(x, y)$, we obtain that there exists an absolute constant $K > 0$ such that for every $t \in [0, T)$,

$$\|z_x(t)\|^2_{L^2} + \int_0^t \|z_{xx}(\sigma)\|^2_{L^2}\,\mathrm{d}\sigma \leqslant K \left[\|(z_0)_x\|^2_{L^2} + \int_0^t \|u_1(\sigma)\|^2_{L^2}\,\mathrm{d}\sigma + \int_0^t \|z(\sigma)z_x(\sigma)\|^2_{L^2}\,\mathrm{d}\sigma\right] . \qquad (8.6)$$

Using the elementary interpolation inequality

$$\|\psi\|_{C[0,1]} \leqslant \sqrt{2}\|\psi\|^{\frac{1}{2}}_{L^2[0,1]}\|\psi_x\|^{\frac{1}{2}}_{L^2[0,1]} \quad (\psi \in H^1_0(0, 1)),$$

it follows that for almost every $t \in [0, T)$,

$$\|z(t)\,z_x(t)\|_{L^2} \leqslant \|z(t)\|_{C[0,1]}\|z_x(t)\|_{L^2} \leqslant \sqrt{2}\|z(t)\|^{\frac{1}{2}}_{L^2}\|z_x(t)\|^{\frac{3}{2}}_{L^2} .$$

From (8.5) we easily obtain (using the Poincaré inequality on $[0, 1]$) that for every $t \in [0, T)$,

$$\|z(t)\|^2_{L^2} + \int_0^t \|z_x(\sigma)\|^2_{L^2}\,\mathrm{d}\sigma \leqslant \|z_0\|^2_{L^2} + \|u_1\|^2_{L^2([0,T];U)} . \qquad (8.7)$$

The last two estimates imply that

$$\|z(t)\,z_x(t)\|^2_{L^2} \leqslant 2\left(\|z_0\|^2_{L^2} + \|u_1\|^2_{L^2([0,T];U)}\right)^{\frac{1}{2}}\|z_x(t)\|^3_{L^2} .$$

Let us denote $M = 2\left(\|z_0\|^2_{L^2} + \|u_1\|^2_{L^2([0,T];U)}\right)^{\frac{1}{2}}$. Inserting the last inequality in (8.6) we obtain that for every $t \in [0, T)$ we have

$$\|z_x(t)\|^2_{L^2} \leqslant K\left[\|(z_0)_x\|^2_{L^2} + \int_0^t \|u_1(\sigma)\|^2_{L^2}\,\mathrm{d}\sigma + M\int_0^t \|z_x(\sigma)\|^3_{L^2}\,\mathrm{d}\sigma\right] .$$

From the above estimate it follows that for every $t \in [0, T)$,

$$\|z_x(t)\|^2_{L^2} \leqslant K_1 + \int_0^t K_2(\sigma)\|z_x(\sigma)\|^2_{L^2}\,\mathrm{d}\sigma , \qquad (8.8)$$

where, for every $t \in [0, T)$,

$$K_1 = K\left[\|(z_0)_x\|^2_{L^2} + \int_0^T \|u_1(\sigma)\|^2_{L^2}\,\mathrm{d}\sigma\right] ,$$

$$K_2(t) = KM\|z_x(t)\|_{L^2} .$$

From (8.7) we see that $K_2 \in L^2[0, T]$, whence $K_2 \in L^1[0, T]$. The estimate (8.8) and Gronwall's inequality yield that

$$\|z_x(t)\|^2_{L^2} \leqslant K_1 \mathrm{e}^{\|K_2\|_{L^1[0,T]}} \qquad (t \in [0, T)) . \qquad (8.9)$$

Thus, $\|z(t)\|_X = \|z_x(t)\|_{L^2}$ remains bounded on $[0, T)$. According to Remark 7.5, this is a contradiction. Hence $T = \infty$. □

Proof of Theorem 8.1. We use the same notation as in Theorem 8.3. Let $z_0 \in X$ and $u_1 \in L^2_{\rm loc}([0,\infty); U_1)$. According to Theorem 8.3 there exist unique functions $z \in C([0,\infty); X)$ and $y \in L^2_{\rm loc}([0,\infty); Y)$ that satisfy (7.5) and (7.6) for all $t \geqslant 0$. By Remark 7.4 z satisfies

$$z(t,x) - z_0(x) = \int_0^t [z_{xx}(\sigma,x) \\ -z(\sigma,x)z_x(\sigma,x) + u_1(\sigma,x)]\,\mathrm{d}\sigma, \quad (8.10)$$

for every $t \geqslant 0$ and the function under the integral is in $L^2_{\mathrm{loc}}([0,\infty);X_{-1})$. Differentiating both sides, we obtain that the first equation in (8.1) holds in $L^2_{\mathrm{loc}}([0,\infty);X_{-1})$. The facts that $z \in C([0,\infty);\mathcal{H}^1_0(0,1))$ and $z \in L^2_{\mathrm{loc}}([0,\infty);\mathcal{H}^2(0,1))$ follow from the well-posedness of $\Sigma^{\mathcal{N}}$, using that $z \in C([0,\infty);X)$ and $y = z \in L^2_{\mathrm{loc}}([0,\infty);Y)$. Consequently it is easy to see that all the terms on the right-hand side of the first equation in (8.1) are in $L^2_{\mathrm{loc}}([0,\infty);L^2[0,1])$. It follows that the first equation in (8.1) holds in $L^2([0,\tau];L^2[0,1])$ and hence $z \in \mathcal{H}^1_{\mathrm{loc}}((0,\infty);L^2[0,1])$. We have thus shown that z satisfies (8.1) and (8.2).

For the uniqueness part, assume that z satisfies (8.1) and (8.2), so that $\dot{z} = Az + B_1u_1 + \mathcal{N}(z,z)$ in $L^2_{\mathrm{loc}}([0,\infty);L^2[0,1])$. According to Proposition 4.6, z is a solution of (7.5), (7.6), which is unique according to Theorem 8.3. □

9 Local well-posedness of the Navier-Stokes system

In this section we show that the abstract nonlinear feedback theory from Section 7 can be used to derive existence and uniqueness of local (in time) strong solutions for the Navier-Stokes system, which describes the motion of an incompressible viscous fluid in a bounded domain in $\mathbb{R}^3$. This result is not new, it has been obtained by Fujita and Kato [25] and references therein, but we think that it is interesting to see a proof based on well-posed system theory. Related recent work using abstract linear systems theory for the analysis of the Navier-Stokes system is in Haak and Kunstmann [31]. The existence of global strong solutions for this system (in $\mathbb{R}^3$) is a famous open problem with a large prize, see the official problem statement by Fefferman [24]. In $\mathbb{R}^2$ the Navier-Stokes system is known to have global strong solutions (we refer again to [25]) and the proof of this fact has many common points with our approach to the Burgers equation in the previous section.

We assume that the fluid occupies a bounded open set $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ of class C^2. The Navier-Stokes system (see, for instance, Sohr [60]) is

$$\rho\dot{z} - \nu\Delta z + \rho(z\cdot\nabla)z + \nabla p = u_1\,,\ t \geqslant 0,\ x \in \Omega, \quad (9.1)$$

$$\operatorname{div} z = 0,\ t \geqslant 0,\ x \in \Omega, \quad (9.2)$$

$$z = 0,\ t \geqslant 0,\ x \in \partial\Omega, \quad (9.3)$$

$$z(0,x) = z_0(x),\ x \in \Omega. \quad (9.4)$$

In the above system the unknown functions are $z(t,x)$, the Eulerian velocity field of the fluid, and $p(t,x)$, the pressure field in the fluid. The given positive constants ρ and ν denote the density and the viscosity of the fluid.

As in the previous section, the function spaces in this section contain only real-valued functions. We introduce

$$\widehat{\mathcal{H}^1}(\Omega) = \left\{ q \in \mathcal{H}^1(\Omega) \ \Big| \ \int_\Omega q(x)\,\mathrm{d}x = 0 \right\},$$

which is a Hilbert space when endowed with the inner product inherited from $\mathcal{H}^1(\Omega)$.

The main result in this section is:

Theorem 9.1 *For every initial state $z_0 \in \mathcal{H}^1_0(\Omega;\mathbb{R}^3)$ with $\operatorname{div} z_0 = 0$ and every input function $u_1 \in L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega;\mathbb{R}^3))$, there exists $T > 0$ and a unique solution $(z,\,p)$ of (9.1)-(9.4) such that*

$$z \in \mathcal{H}^1((0,T);L^2(\Omega;\mathbb{R}^3)) \cap C([0,T];\mathcal{H}^1_0(\Omega;\mathbb{R}^3)) \\ \cap L^2([0,T];\mathcal{H}^2(\Omega;\mathbb{R}^3)), \quad (9.5)$$

$$p \in L^2([0,T];\widehat{\mathcal{H}^1}(\Omega)), \quad (9.6)$$

and the equation in (9.1) holds in $L^2([0,T];L^2(\Omega;\mathbb{R}^3))$.

Moreover, if $[0,\delta)$ is the maximal interval of existence of the solution of (9.1)-(9.4), then for every $c > 0$ there exists $T \in [0,\delta)$ such that $\|z(T)\|_{\mathcal{H}^1_0(\Omega;\mathbb{R}^3)} \geqslant c$.

To prove the above theorem we need some notation. Consider the Hilbert space

$$L^2_\sigma(\Omega) = \{\varphi \in L^2(\Omega;\mathbb{R}^3) \mid \operatorname{div}\varphi = 0,\ \ \varphi\cdot n = 0 \ \text{on}\ \partial\Omega\}.$$

The condition $\varphi \cdot n = 0$ on $\partial\Omega$ should be understood in the weak sense:

$$\int_\Omega \varphi\cdot\nabla g\,\mathrm{d}x = 0 \qquad \forall\, g \in \mathcal{H}^1(\Omega).$$

It is easily checked that $L^2_\sigma(\Omega)$ is a closed subspace of $L^2(\Omega;\mathbb{R}^3)$, so that it makes sense to consider the orthogonal projector P of $L^2(\Omega;\mathbb{R}^3)$ onto $L^2_\sigma(\Omega)$. This is called the *Leray projector* or the *Helmholtz projector*, see for instance [60, p. 82], and it is often used to eliminate the pressure from (9.1)-(9.4). Denote

$$G(\Omega) = (I-P)L^2_\sigma(\Omega). \quad (9.7)$$

It is a well-known result (see for instance [60, Section II.2.5]) that $G(\Omega) = \nabla(\widehat{\mathcal{H}^1}(\Omega))$. It is easy to see that ∇ is one-to-one between these spaces. By the closed graph theorem ∇ has a bounded inverse

$$\mathcal{M} \in \mathcal{L}(G(\Omega),\widehat{\mathcal{H}^1}(\Omega)). \quad (9.8)$$

An important role in the remaining part of this section will be played by the *Stokes operator on* Ω, denoted by A_0, which is defined by

$$A_0\varphi = -\frac{\nu}{\rho}P\Delta\varphi,$$

$$\mathcal{D}(A_0) = L^2_\sigma(\Omega) \cap \mathcal{H}^1_0(\Omega;\mathbb{R}^3) \cap \mathcal{H}^2(\Omega;\mathbb{R}^3).$$

It is known, see for instance [60, Ch. III, Theorem 2.1.1], that A_0 is a strictly positive operator on $L^2_\sigma(\Omega)$. As usual, we denote

$$H = L^2_\sigma(\Omega), \qquad H_1 = \mathcal{D}(A_0),$$

$$H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}}), \qquad H_{\frac{3}{2}} = \mathcal{D}(A_0^{\frac{3}{2}}).$$

According to [60, Ch. III, Lemma 2.2.1],

$$H_{\frac{1}{2}} = \left\{\varphi \in \mathcal{H}^1_0(\Omega;\mathbb{R}^3) \;\mid\; \operatorname{div}\varphi = 0\right\}.$$

We introduce a well-posed linear system Σ^P as follows. We introduce the state, input and output spaces

$$X = H_{\frac{1}{2}}, \quad U_1 = U_2 = L^2(\Omega;\mathbb{R}^3), \quad Y = H_1,$$

and the operator $A : \mathcal{D}(A) \to X$ by

$$A\varphi = -A_0\varphi, \qquad \varphi \in \mathcal{D}(A) = H_{\frac{3}{2}}. \tag{9.9}$$

Since $A < 0$, A generates an analytic operator semigroup $\mathbb{T}$ on X. As in Section 8, the corresponding space $X_{\frac{1}{2}} = \mathcal{D}((-A)^{\frac{1}{2}})$ (endowed with the graph norm of $(-A)^{\frac{1}{2}}$) and its dual with respect to the pivot space X are

$$X_{\frac{1}{2}} = H_1, \quad X_{-\frac{1}{2}} = H.$$

We take $B_k \in \mathcal{L}(U_k, X_{-\frac{1}{2}})$ defined by $B_1 = B_2 = P/\rho$ and, as in Section 8, Φ^k_t, with $k \in \{1,2\}$, are the corresponding input maps.

We denote by C the identity operator of X, which can be restricted to an unbounded observation operator $C \in \mathcal{L}(X_1, Y)$. Now the operators Ψ_∞, Ψ_τ, $\mathbb{F}^k$, with $k \in \{1,2\}$, $\Phi = [\Phi^1\ \Phi^2]$ and $\mathbb{F} = [\mathbb{F}^1\ \mathbb{F}^2]$ are defined exactly as in Section 8.

The analogue of Proposition 8.2 is the result below, with the same proof. We omit to write the analogue of (8.3), as it will not be needed.

Proposition 9.2 *We have that* $\Sigma^P = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ *is a regular linear system.*

With the above notation, we show below that the system (9.1)–(9.4) has the form (7.5)–(7.6), where the bilinear continuous operator $\mathcal{N} : X \times Y \to U_2$ is defined by

$$\mathcal{N}(z,y) = -P[(z\cdot\nabla)y]. \tag{9.10}$$

Therefore, the abstract version of Theorem 9.1 is the following:

Theorem 9.3 *With the specified spaces and operators, the system defined by* (7.5), (7.6) *is locally well-posed.*

The proof below follows the ideas in Takahashi and Tucsnak [70]. Similar techniques could be used to prove the local well-posedness of the system describing a spherical rigid body moving in an incompressible viscous fluid occupying all the remaining space (this was the original motivation in [70]).

Proof. In order to apply Theorem 7.6 it suffices to show that condition (7.7) holds. To this end we note that for every $z \in X$ and $y \in X_{\frac{1}{2}}$ we have, using the Hölder inequality for three factors, that for any $i \in \{1,2,3\}$,

$$\begin{aligned}
&\int_\Omega z_i^2(x)\left[\frac{\partial y_j}{\partial x_i}(x)\right]^2 \mathrm{d}x \\
&= \int_\Omega z_i^2(x)\left[\frac{\partial y_j}{\partial x_i}(x)\right]^{2/5}\left[\frac{\partial y_j}{\partial x_i}(x)\right]^{8/5} \mathrm{d}x \\
&\leqslant \|z_i^2\|_{L^{5/2}(\Omega)} \left\|\left[\frac{\partial y_j}{\partial x_i}\right]^{2/5}\right\|_{L^5(\Omega)} \left\|\left[\frac{\partial y_j}{\partial x_i}\right]^{8/5}\right\|_{L^{5/2}(\Omega)} \\
&= \|z_i\|^2_{L^5(\Omega)} \left\|\frac{\partial y_j}{\partial x_i}\right\|^{2/5}_{L^2(\Omega)} \left\|\frac{\partial y_j}{\partial x_i}\right\|^{8/5}_{L^4(\Omega)}.
\end{aligned}$$

Using twice the continuous embedding $\mathcal{H}^1(\Omega) \subset L^p(\Omega)$ for $1 \leqslant p \leqslant 6$ (see, for instance, Adams [2, p. 97]) we obtain that there exists a constant $K > 0$ (depending only on Ω) such that for every $z \in X$ and $y \in X_{\frac{1}{2}}$,

$$\int_\Omega z_i^2(x)\left[\frac{\partial y_j}{\partial x_i}(x)\right]^2 \mathrm{d}x \leqslant K\|z_i\|^2_{\mathcal{H}^1_0(\Omega)}\|y_i\|^{2/5}_{\mathcal{H}^1_0(\Omega)}\|y_i\|^{8/5}_{\mathcal{H}^2(\Omega)}.$$

We have thus proved that there exists $\widetilde{K} > 0$ (depending only on Ω) such that for every $z \in X$ and $y \in X_{\frac{1}{2}}$,

$$\|(z\cdot\nabla)y\|_{U_2} \leqslant \widetilde{K}\|z\|_X\|y\|_X^{1/5}\|y\|_Y^{4/5}. \tag{9.11}$$

From the above estimate and (9.10) it follows that the condition (7.7) holds with $p = 4/5$. Thus by Theorem 7.6 the statement follows. □

Proof of Theorem 9.1. We use the same notation as in Theorem 9.3. Let $z_0 \in X$ and $u_1 \in L^2_{\mathrm{loc}}([0,\infty); U_1)$. According to Theorem 9.3 there exist $T > 0$ and unique functions $z \in C([0,T];X)$ and $y \in L^2([0,T];Y]$ that satisfy (7.5) and (7.6) for all $t \in [0,T]$. By Remark 7.4 z satisfies

$$\begin{aligned}
&z(t,x) - z_0(x) \\
&= \int_0^t \left[Az(\sigma) - P[(z(\sigma)\cdot\nabla)z(\sigma)] + \frac{1}{\rho}Pu_1(\sigma)\right]\mathrm{d}\sigma,
\end{aligned}$$

for every $t \in [0,T]$ and the function under the integral is in $L^2([0,T];X_{-1})$. Differentiating both sides, we obtain

$$\dot{z}(t) = Az(t) - P[(z(t)\cdot\nabla)z(t)] + \frac{1}{\rho}Pu_1(t) \tag{9.12}$$

holds in $L^2([0,T];X_{-1})$. The facts that $z \in C([0,T];X)$ and $z \in L^2([0,T];Y)$ follow from the local well-posedness of $\Sigma^{\mathcal{N}}$, using that $z \in C([0,T];X)$ and $y = z \in L^2([0,T];Y)$. Consequently it is easy to see that the first and third terms on the right-hand side of in (9.12) are in $L^2([0,T],H)$. The fact that the same is true for the second term follows from (9.11). It follows that (9.12) holds in $L^2([0,T];H)$ and hence $z \in \mathcal{H}^1((0,T);H)$.

It follows from (9.12) that

$$\begin{aligned}\rho\dot{z}(t) &= \nu\Delta z(t) - (z(t)\cdot\nabla)z(t) + u_1(t)\\ &\quad - (I-P)[\nu\Delta z(t) - (z(t)\cdot\nabla)z(t) + u_1(t)] \end{aligned} \tag{9.13}$$

holds with each term in $L^2([0,T];L^2(\Omega;\mathbb{R}^3))$. Hence

$$(I-P)[\nu\Delta z - (z\cdot\nabla)z + u_1] \in L^2([0,T];G(\Omega)),$$

where $G(\Omega)$ has been defined in (9.7). Using $\mathcal{M}$ from (9.8) we define the function

$$p(t) = \mathcal{M}(I-P)[\nu\Delta z(t) - (z(t)\cdot\nabla)z(t) + u_1(t)],$$

so that $p \in L^2([0,T];\widehat{\mathcal{H}^1}(\Omega))$. Then (9.13) becomes (9.1) for $t \in [0,T]$.

For the uniqueness part, assume that z and p are functions satisfying (9.5), (9.6), together with (9.1)-(9.4). By applying the projector P to (9.1) it follows that $\dot{z} = Az + B_1u_1 + \mathcal{N}(z,z)$ in $L^2([0,T],L^2_\sigma(\Omega))$. According to Proposition 4.6, z is a solution of (7.5), (7.6), which is unique according to Theorem 9.3. The uniqueness of p follows by applying the operator $\mathcal{M}(I-P)$ to (9.1). $\square$

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